LABORATORY HANDBOOK OF STATISTICAL METHODS

Graphic and Mathematical Methods

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PREFACE

Many problems of modern business are statistical in nature. The business man, when trying to find the solution of these problems, usually can indicate the *form* of result or summary which he desires. Research workers over the last few years have developed a *statistical procedure* which should enable the business man to see the statistical technique required for his particular purpose. In order that the statistical technique may be of value to the business man, a coordination between pure methodology and problem is indispensable.

In adapting the methods of the research worker to the needs of business problems, the essential requirements are: first, that the technical terms of the analytical processes be made clear both as to their meaning and as to their use; second, that the standard mathematical processes be set up in such a manner that the form in which they appear may be used to work out a variety of problems. The authors recognize fully that to many it may seem that these requirements can be met only by a complete text of statistical methods. They have chosen, however, to emphasize the significance of the method rather than details of it.

The statistical summaries which the business man may want in connection with a specific problem may be presented either in graphical or in numerical form. The division of this text follows a similar plan. Book I consists of a description of how to construct graphs for business purposes. Book II outlines the purpose of certain statistical methods, together with the solution of numerical problems so that the form in which the method is carried through may be followed. In both parts of the text the major portion of the discussion has been devoted to the practical application of the processes which are involved. In this way the technical processes are coordinated with their use in actual problems.

The scope of the text has been limited to those elementary methods which are most commonly used. The exception to this is the chapter on probability paper which has been introduced because it presents a simple procedure sufficiently accurate for

most business problems. The use of probability paper for graduating frequency distributions is the counterpart of the common methods of fitting trend lines to data. Since the authors have desired to restrict the book to an outline of procedures, no examples have been suggested for the second part of the text.[1] The present edition of the "Laboratory Handbook" is a result of a thorough revision of the previous editions which have been used at the Harvard Business School for a number of years. Before asking the student to use a particular method, many teachers of statistics are accustomed to present a review of the method in connection with some example. If such a procedure is followed, the illustrations worked out in the tables should provide a guide for clarifying many misunderstandings that a student may have. The text will enable one to see how the method may be tied in with actual problems.

Professors D. H. Davenport and J. W. Horwitz have made many suggestions in regard to the revision of the text for which the authors take pleasure in acknowledging their indebtedness. Other present and former members of the staff have contributed largely in the writing of earlier editions as well as the present one. The assistance of Miss E. H. Puffer in reading the proof also is acknowledged. The necessary typing and secretarial work has been handled ably by Miss G. E. Crockett and Miss A. C. MacLaughlin.

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August, 1931.

[1] For lists of numerical problems see Mills and Davenport, *Problems and Tables in Statistics*, or Chaddock and Croxton, *Exercises in Statistical Methods*.

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BOOK I

GRAPHIC METHODS

LABORATORY HANDBOOK OF STATISTICAL METHODS

CHAPTER I

BASIC PRINCIPLES

There are two ways in which statistical information may be presented. These are (1) by the use of numerical tables as described in the Appendix of Book II, and (2) by the use of some pictorial form of presentation. The presentation of statistical information in the form of graphs is by no means new. With the increasing use of statistical information, however, there has been a corresponding growth in the use of graphs. In 1786 William Playfair published in London a commercial and political atlas in which numerous colored graphs appeared. Many of the graphs presented for the first time in this book are identical in principle with those used today, so that the art of graphic presentation of statistics is at least 150 years old. One of Playfair's graphs is reproduced in Fig. 1.

Fundamentally, there are two types of graphs which the statistician may desire to use. The purpose of the first type is to present statistical ideas in a pictorial form. The purpose of the second type is to provide a calculating device, which the engineer considers a working drawing. Additional information may be obtained from such a drawing. By far the larger number of business or statistical graphs drawn are included in the first type. Because of the importance of the first type, the construction of most of the graphs described in the subsequent chapters are of this type.

If we limit ourselves to the first type, we may state the basic idea as a twofold proposition. A graph should be drawn either when it will picture facts which cannot otherwise be presented or

when it will present facts in a better way than can otherwise be done.

With this in mind, the ideal graph should present information in a simple, clear, complete, and truthful manner. Simplicity is of primary importance in conveying an idea, since the effect of the graph is lost if it is complicated. Clearness means that the information must be presented so that there can be no doubt in regard to the correct interpretation of the data. Completeness

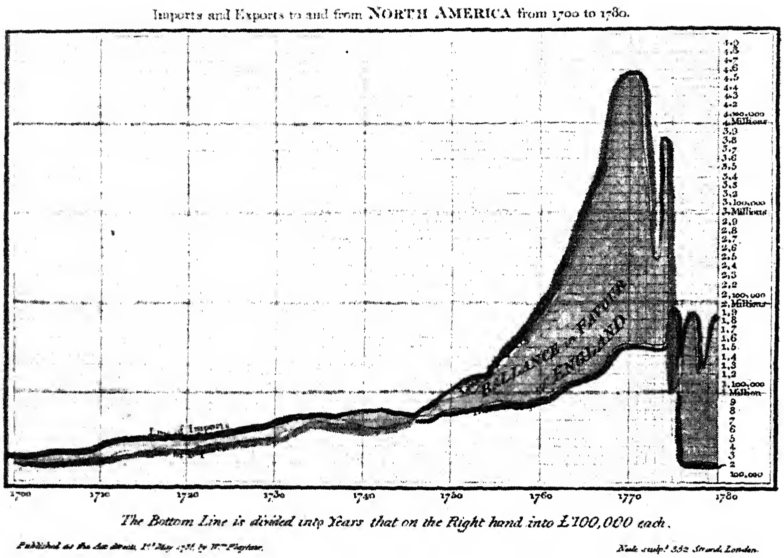


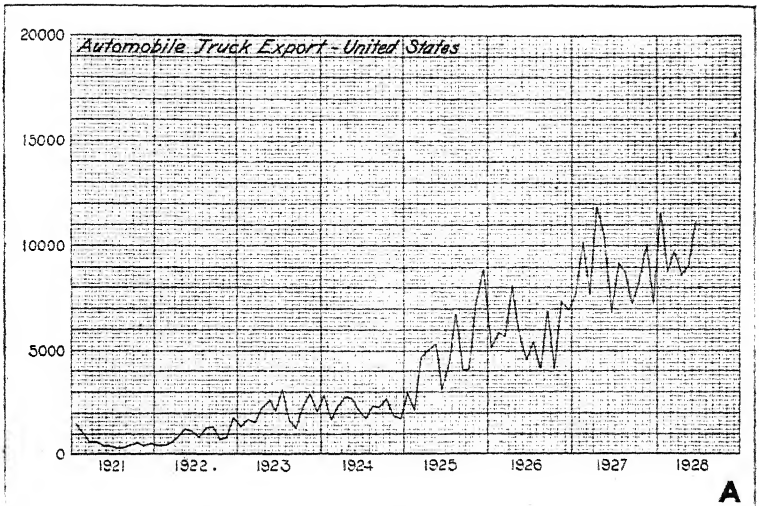
FIG. 1.

is essential because the presentation of facts is of value only when all the basic ideas in regard to those facts are easily available. Finally, a graph must be truthful; for otherwise, distorted or misleading ideas will result from its interpretation.

An illustration of some of the problems in graphical presentation may be found in the Hibben Truck Company case.

HIBBEN TRUCK COMPANY

The sales manager of an American automobile truck manufacturing company prepared a graph, on a commercially printed grid, to show the total export of automobile trucks from the


Automobile Truck Export - United States
20000
15000
10000
5000
0
1921
1922.
1923
1924
1925
1926
1927
1928
A

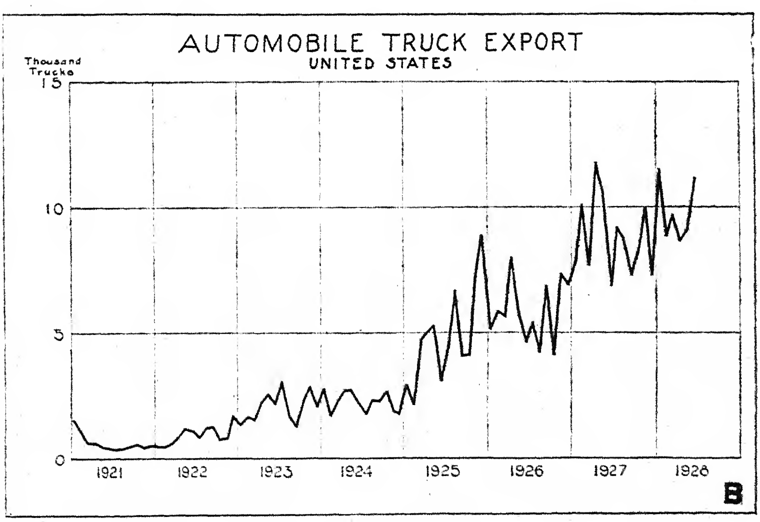

AUTOMOBILE TRUCK EXPORT
UNITED STATES
Thousand Trucks
15
10
5
0
1921
1922
1923
1924
1925
1926
1927
1928
B

FIG. 2.

United States by months. The data covered the period, January, 1921 to June, 1928. These were shown in Graph A.

His purpose in having this graph prepared was that he might plot the export sales of his company below the curve representing the total truck export for the country, and thus determine whether it seemed worth while to attempt to increase his company's export business. He intended to show the graph to the general manager in order to help secure authorization to increase the foreign sales budget.

On looking at the graph the sales manager was impressed with the fact that the curve was somewhat obscured by the multitude of lines comprising the background, or grid, of his copy. Because he thought that the trend, or general movement, of the line representing the data was the important factor toward which the attention of the general manager should be focused, he conceived the idea of preparing his own grid on which all but a few of the background lines were omitted and at the same time the width of the plotted curve was increased. A graph was prepared with this idea in mind. This is shown in Graph B.

The Hibben Truck Company case illustrates some of the objections frequently found in graphs. Two graphs are shown, the original, A, and a revised graph, B. Some of the deficiencies which appear in A and which are corrected in B will now be pointed out.

In the first place, the printed background, which was used in Graph A, presents too heavy a network of grid lines. This network naturally catches the eye as the important thing. In the second place, the plotted line does not stand out distinctly, since the fine grid lines cover the drawing of the light curve. In addition to this, the base line as shown is of the same width as certain other lines of the grid. This line, however, should always be heavier than all others in this type of grid.

The title, which is printed on the grid, should be placed above the grid, as on Graph B. Furthermore, since the words "United States" are a sub-title, they should be placed in smaller letters below the main title. Next, it will be noted that the scale caption *thousand* in Graph B eliminates the necessity for a large number of ciphers in Graph A.

It is to be questioned also whether for the solution of the problem the arithmetic or semilogarithmic scale should be used. This question is discussed in Chapter IV.

The printed grid luckily just accommodated the data to be plotted. Often printed backgrounds are either too long or too short for use in connection with a given series of data. The monthly values were plotted in the middle of the spaces rather than on the line, as is frequently done. This certainly was in accordance with good practice. The time-scale numbers were placed at the bottom of the graph as they should have been located. Letters or words indicating each month also were omitted. This omission is usually desirable in a graph of this type because insignificant details tend to draw the eye from the significance of the plotted line.

In the foregoing paragraphs the underlying principles of graphics have been described from the point of view of the first type of graph, that is, one which is intended to present a picture of statistical information. Illustrations of the second type of graph, which are to be used as working drawings as well as pictures, appear occasionally in the following chapters.

CHAPTER II

CONSTRUCTION PRINCIPLES

PAPER AND DRAWING BOARD

The paper on which the drawing is to be made should be carefully selected. Usually the standard typewriter size 8½ by 11 inches is most satisfactory, since it is convenient for binding in a report with typed sheets. Unruled paper should be used since the background or grid for each graph is drawn in accordance with the principles stated in Chapter I.

A small drawing board about 12 by 18 inches is convenient to use. The T square should be placed upon the board in such a manner that the head of the T is along the left-hand edge of the board. Next, the paper should be fastened to the board by thumb tacks[1] in such a way that the lower edge is parallel to the upper edge of the T square which extends across the board. Practically, this may be accomplished by setting the T square at a low position on the board and then placing the paper so that its lower edge rests against the upper edge of the blade of the T square. The paper is fastened most conveniently by two thumb tacks in the upper corners of the paper. This leaves the paper flat so that in drawing lines the T square will slide over the entire surface, and yet be in contact with the paper.

One of the first things that will be needed in constructing the grid is a statistician's scale. The scale recommended is a boxwood scale with a triangular cross section, which is similar to those used by engineers. Since the triangular scale has six faces, there is space for six different scales. One of the six different faces is provided with three logarithmic scales so that there are actually eight scales. Five of the scales are denoted by numbers. These are 20, 30, 40, 50, and 60, indicating the number of divisions to the inch into which each is divided. These various divisions of the inch are grouped and numbered by tens.

[1] A special adhesive tape which serves the same purpose is manufactured by the Eugene Dietzgen Company under the trade name "Scotch Holdfast Drafting Tape."

CONSTRUCTING THE GRID

Figure 3 illustrates the layout of the grid. It will be noticed that the figure contains both black and red lines. The *red lines* indicate pencil construction lines which are to be erased from the finished drawing. In making pencil lines, a pencil which will draw light lines easily erased should be used. Care should be taken not to press the pencil too heavily on the paper, for pressure has the effect of indenting lines which cannot be removed even though the graphite marks are erased.

The steps to be taken in constructing the grid will now be enumerated. The first is to draw a vertical line for the right-hand edge of the grid ½ inch from the right-hand edge of the paper, line *A*, Fig. 3. This coincides with the right-hand margin line. In practice the drawing of the vertical line is accomplished by the use of a triangle. One leg of the triangle is placed against the T square blade. Since the edge of the T square lying across the board is parallel with the bottom of the paper and since the triangle has one angle of 90°, the other leg of the triangle will always be perpendicular to the lower edge of the paper. Because the triangle may be moved along the upper edge of the horizontal arm of the T square, vertical lines in any location may be drawn. By this means the vertical line determining the right-hand margin is drawn. The margins which are to define the limits beyond which no part of the graph should be constructed should, as is indicated in Fig. 3, be set at ½ inch from the right, left, and lower edges of the paper and 1¼ inches from the edge where the punched holes for binding are located. That is, ¾ inch is allowed for the binding and ½ inch for the ordinary margin. The next step is to draw a horizontal grid line about 1¼ inches from the bottom of the paper, line *B*, Fig. 3. This means that if the margin is to be ½ inch, then ¾ inch will be left for descriptive scale marking, a key or legend, and a statement of the source of the data. When there is a large amount of statistical material, this descriptive space should be made wider. However, a good working rule is 1¼ inches from the bottom of the paper. The horizontal grid line should be 9 inches long as measured from the right-hand boundary line first drawn. It will be found that 9 inches is a convenient length for practically all graphs which have the longer axis horizontal.

The next thing to determine is the height of the grid. This should be approximately 4½ inches, but will vary according

to the range of the data and the scale chosen. A suitable scale for plotting the data can be chosen from those on the statistician's scale. The height and the length of the grid are now determined so that the grid can be enclosed. Sufficient vertical lines are next drawn to allocate spaces for the plotting of the data. In the case of time series graphs, space to complete a certain number of full years is allowed, even though the data may be available for only a portion of the last year to be plotted.

In many cases it will be found that no one of the statistician's scales applied directly will subdivide either the horizontal length or the vertical height of the grid in such a way as to give the desired number of divisions. The scale subdivisions for the desired length or height of the grid then must be obtained by either reduction or enlargement of some convenient scale. The two cases just enumerated will be illustrated by the subdivision of a vertical distance. Obviously, by turning the scale through 90°, the horizontal distance can be dealt with similarly.

The method of scale reduction is shown in Fig. 4. In it are to be found four illustrations. The two upper ones apply to scale reduction, and the two lower apply to scale enlargement. In scale reduction the selected scale is longer than the vertical line to be divided. The vertical height is represented by the distance AB. Assume, as in the illustration, that line AB is to be divided into 10 units. If the zero of the statistician's scale is placed on the line AC, and the 10 on a parallel line, BD, then the points corresponding to 1, 2, 3, etc., can be marked. The blade of the T square is used to draw horizontal lines through these points which will cut the line AB in the 10 desired subdivisions. This description applies either to an arithmetic scale or to a logarithmic scale.

Assume that it is desired to enlarge a scale as in the lower two illustrations of Fig. 4. We desire to divide the line AB into 10 equal divisions, but our scale with 10 divisions is shorter than the line AB. Select some point arbitrarily such as E. From E draw the line BE. At some point along the line BE, the 10 on the statistician's scale can be placed so that the edge of the scale is parallel to AB with the zero of the scale on the line AE. After the scale has been so placed, the points corresponding to the scale divisions 1, 2, 3, etc., can be marked. Then lines radiating from the point E can be drawn through these points, which will project the 10 divisions desired on the line AB. This description applies either to the arithmetic or to the logarithmic scale.

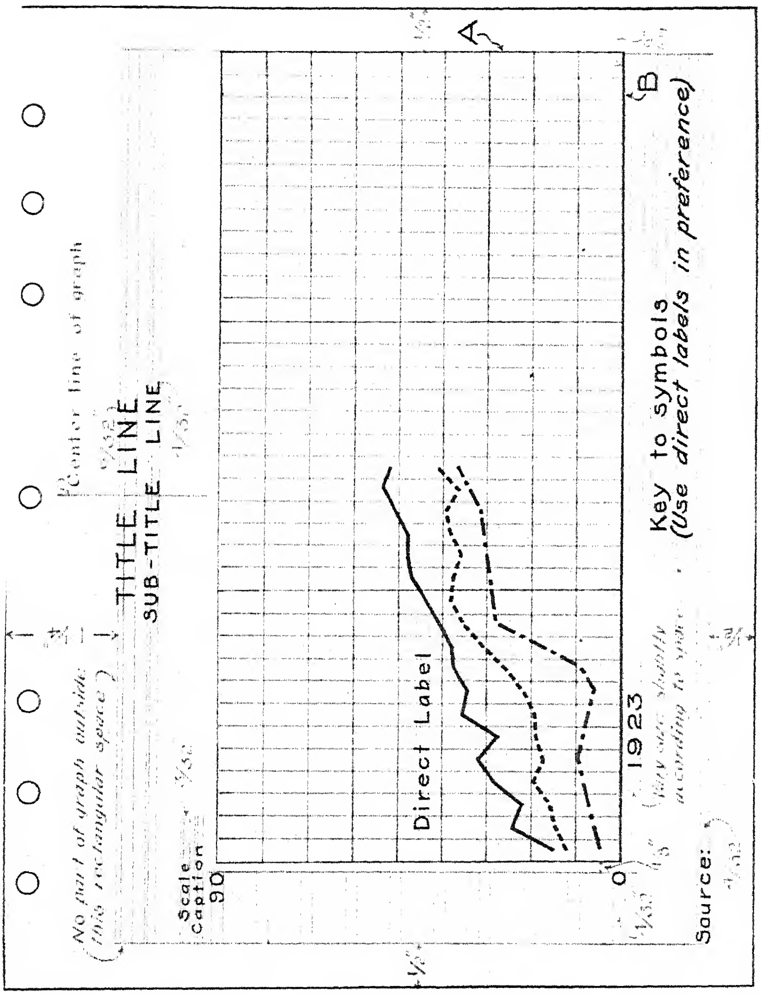

No part of graph outside
this rectangular space
Center line of graph
TITLE LINE
SUB-TITLE LINE
Scale
caption
90
0
1923
Direct Label
A
B
Key to symbols
(Use direct labels in preference)
Source:

FIG. 3.

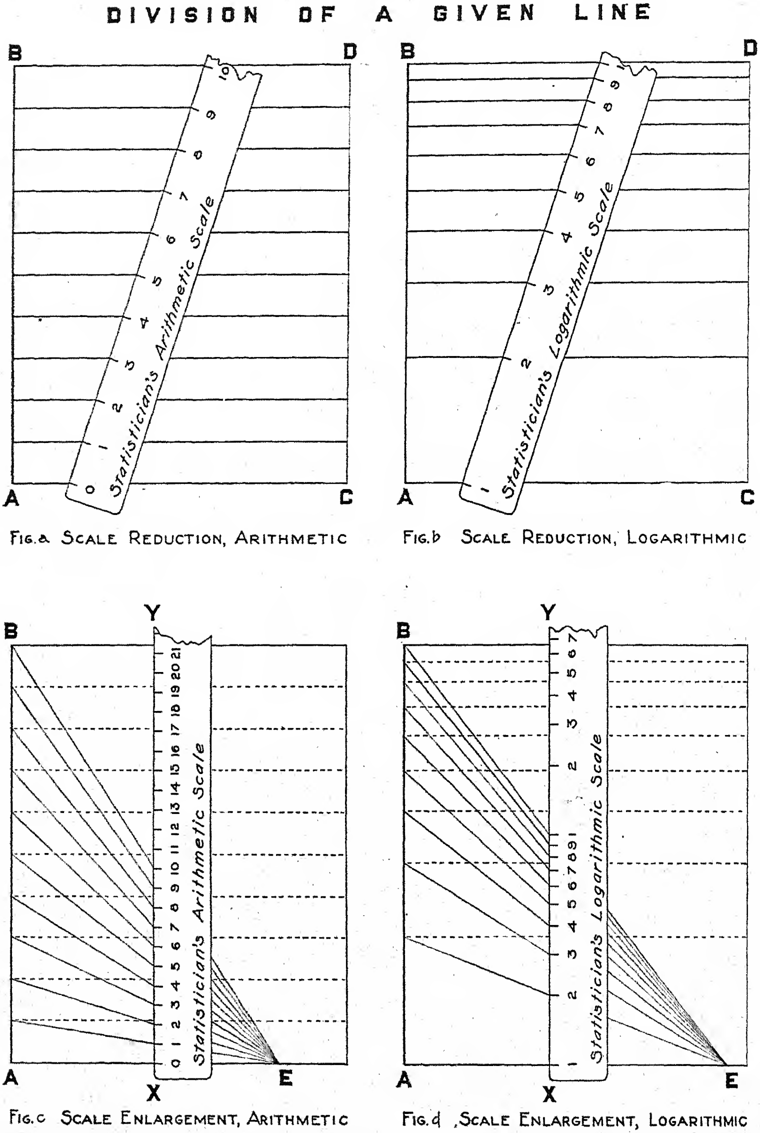

DIVISION OF A GIVEN LINE
B
D
A
C
Statistician's Arithmetic Scale
Fig. a Scale Reduction, Arithmetic
B
D
A
C
Statistician's Logarithmic Scale
Fig. b Scale Reduction, Logarithmic
B
Y
A
X
E
Statistician's Arithmetic Scale
Fig. c Scale Enlargement, Arithmetic
B
Y
A
X
E
Statistician's Logarithmic Scale
Fig. d Scale Enlargement, Logarithmic

FIG. 4.

Next, the data are plotted. For most series, the horizontal position of the plotted point is taken as the midpoint between the boundary lines for the time unit used. This can be determined by a set of temporary vertical lines drawn through the midpoints between the boundary lines, or the midposition can be located carefully by eye. In the following chapters one or two exceptions to this general rule will be found. These exceptions occur in the cases of the cumulative graph and the zee graph. The vertical height of the plotted point is determined in each case by a direct reading from the statistician's scale selected. The points as plotted are connected by light, straight pencil lines, which are inked later.

Since it is desirable that only a few horizontal and vertical lines appear in the finished graph, the lines which are to be inked should be identified when laying out the graph.

Because the graph represents a structure resting upon a base or foundation from which vertical values are read, vertical scale numbers are placed at the left boundary of the field or grid. The vertical scale is incomplete without a caption to indicate the kind of unit of measurement, such as "dollars," "tons," "thousands of square feet." This scale caption must be exact and clear. For example, price may be measured in units of dollars and cents, yet this designation in the caption may be incomplete without the inclusion of "per ton," "per pound." Likewise, "per cent" in a caption may be incomplete unless we designate it as a per cent of something, as "per cent of normal," or "per cent of total."

Economy of effort and a more pleasing appearance may be gained by the omission of ciphers in the scale numbers, provided the proper transposition is indicated in the caption. For instance, the numbers "500," "1,000" may read "5," "10," with the caption "hundreds of tons." When such a contraction is made, it should be in terms of a convenient division such as hundreds, thousands, millions, or their combination. The scale caption should be centered above the column of scale numbers. Individual scale numbers should be centered opposite the grid lines to which they refer. This may be accomplished easily by either guide line device shown in Fig. 5 or Fig. 7.

The horizontal scale is placed below the grid. When this scale is used to indicate time values, such as dates, the meaning is obvious so that no scale caption need be used. On the other hand, if quantities are used for the horizontal scale, as is necessary

in correlation graphs, a caption should be placed on a horizontal line centered immediately below the scale values (see page 63).

An important part of graph construction is the selection of an appropriate title. As a rule a title is adequate if it answers part or all of the three questions: What, where, and when? For example, "Pig Iron Production" indicates what; "United States" signifies where; and "Monthly, 1924" tells when. The title may be divided into two or more lines with a main title containing the most important information, such as "what," and subtitles containing supplementary facts such as "where" and "when." The title should be brief and should not contain a statement of the interpretation which the graph is designed to show. Titles should be centered with respect to the overall limits of the entire drawing.

When several plotted lines are included, they may be differentiated by colors or by variations of a black line such as continuous. solid, or dashes and dots. Differences in the width of plotted lines should not be used unless it is intended to convey that the line of lesser width has a subordinate value. The lines should preferably be identified by a direct label, but a key may be used, when necessary. These labels should be horizontal and near the left-hand end of the line. Arrows may be used if identification otherwise would be difficult. Indirect identification is secured by means of a legend or key of symbols located preferably just below the grid. Areas are distinguished by colors or by cross-hatching. For example, see Fig. 14, component column graph. In addition, the source should be indicated on the graph so that data on which the graph is based may be located and credited.

Two devices to establish guide lines for lettering are illustrated, with directions as to use, on pages 15, 17. The size of the letters follows no strict rule, save that they ought to be sufficiently large to balance well with the rest of the drawing. Average sizes, which, on the whole, will be found very satisfactory, are indicated in Figs. 3 and 6.

After the drawing has been completed, the lines are inked by means of a ruling pen. Pens should be cleaned occasionally by washing them in water and wiping with cloth to prevent rusting. When the ruling pen is used, care must be taken to hold the pen exactly vertical. The thickness of the sides of the pen keep the inked line slightly away from the triangle. If, however, the point of the pen is tipped inward a little, the fresh ink on the line is likely to flow under the triangle leaving a blot on the work.

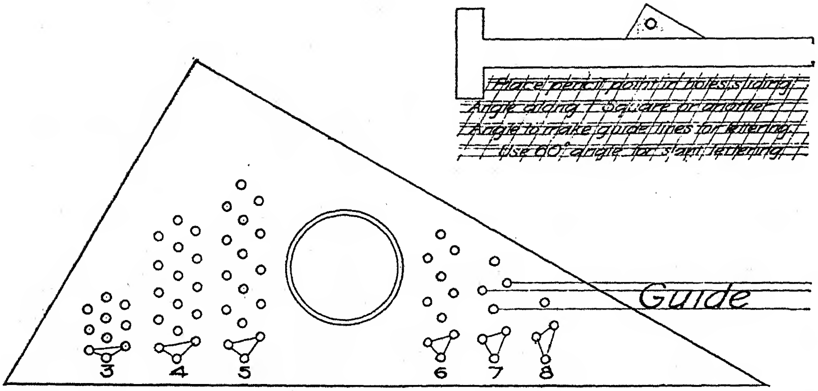


FIG. 5.

Description of Lettering Angle

The Lettering Angles are designed to give a *quick* and *easy* method of making *accurately* spaced guide lines for lettering drawings, etc.

The Lettering Angle is designed to slide on the hypotenuse when standard spacings are desired. However, either of the other two sides may be used for other spacings.

The holes in each column, beginning at the bottom, are arranged in groups of three. This enables the drawing of three guide lines for each line of lettering when it is desired to use both lower case and capital letters. The lowest hole of a group provides for the bottom guide line of the letters; the highest hole of the group gives the height of the capitals; the middle hole gives the standard height of the lower case letters, which is two-thirds the height of the capitals. The groups themselves are so arranged that the spacing between them is two-thirds the height of the capitals.

The figure under each column of holes denotes the height of capital letters in thirty-seconds of an inch.

To use:

Place the point of a 2H, 3H, or 4H pencil through a hole in the desired group and slide the angle along the T square or another triangle; then place the pencil point through another hole and slide back. The guide lines will be very accurately spaced, and drawn much more rapidly than by laying off with scale or dividers. The holes are tapered so as to prevent the breaking of the pencil point.

If you already have established a standard spacing for your lettering, other than given direct on the angle, you can locate holes that will give you spacing as follows: Lay out the lines of your standard spacings; set some hole in the Lettering Angle over the bottom line and mark the holes that coincide with the other lines, so as to distinguish them easily. Equally spaced holes can be obtained by this method also.

Equally spaced lines can be obtained by using a similar hole in each group, and these can be divided in ½, ⅔, or ¾, by dividing one space, then using again a similar hole in each group.

By using one of the other sides and all holes in a column, various spacings for bills of material can be obtained, giving the lettering spacing as well as the spacing between.

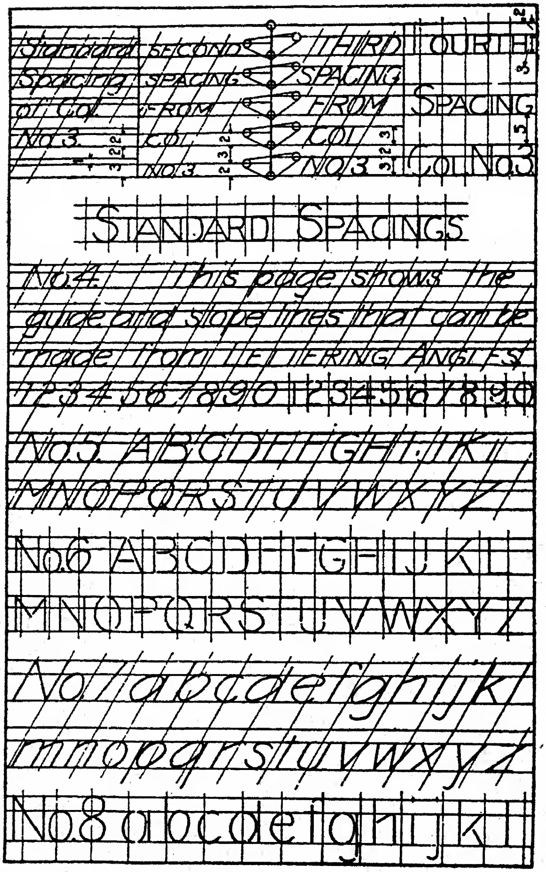

Standard Spacing of Col No 3
SECOND SPACING FROM COL No 3
THIRD SPACING FROM COL NO 3
FOURTH SPACING COL No 3
STANDARD SPACINGS
No 4 This page shows the guide and slope lines that can be made from LETTERING ANGLES
1234567890 1234567890
No 5 ABCDEFGHIJKL MNOPQRSTUVWXYZ
No 6 ABCDEFGHIJKL MNOPQRSTUVWXYZ
No 7 abcdefghijkl mnopqrstuvwxyz
No 8 abcdefghijkl

Fig. 6.

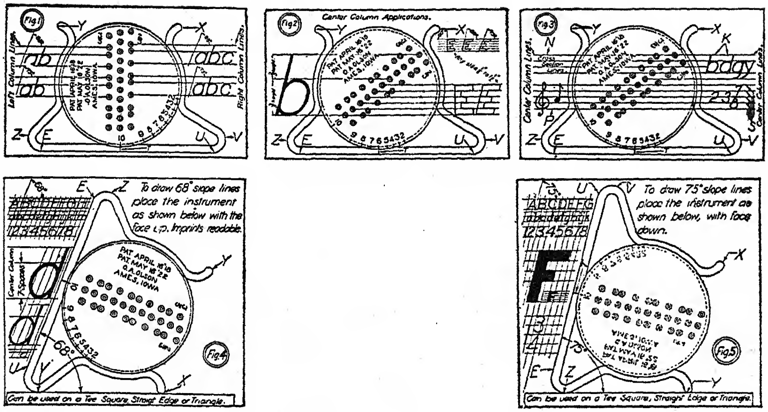


FIG. 7.

Instructions

Sharpen one end of a 6H drawing pencil so that the lead exposed has a sharp conical point approximately five-sixteenths of an inch long.

Use the T square in the same manner as in drawing. Place the instrument on the drawing board so that the base bar marked U-E in Fig. 1 will rest on the upper edge of the T square. Have the readable side up.

The fraction $\frac{3}{5}$ at the top of the disc indicates that holes in the column to the right are so spaced that the ratio of the distances between the guide lines will be $\frac{2}{5}$ and $\frac{3}{5}$ of the total height of a capital letter of that system. This is true for any position of the disc. Study Fig. 1. This is the ratio usually used by civil engineers.

The fraction $\frac{2}{3}$ indicates that the ratio will be $\frac{1}{3}$ and $\frac{2}{3}$ of the total height of the letter. This is the ratio used in the REINHARDT system.

The numbers 10, 9, 8, 7, 6, 5, 4, 3, 2 are numerators of fractions whose understood denominators are 32. If the disc is turned so that the number 6 is directly above the mark T on the base bar, the capital letters in each of the three systems of guide lines will be $\frac{6}{32}$ of an inch high.

How to Draw Lines

Assume that you have placed the disc in the above-mentioned position. Place the conical 6H drawing pencil point in the *second* hole from the top. Hold the pencil in a plane that is perpendicular to the paper but incline the pencil slightly toward the direction that you are drawing.

After the pencil point is in place, press the point against the side of the hole so as to keep the base bar in sliding contact with the T square.

By means of the pencil point slide the instrument along the edge of the T square until it arrives at the right terminal for the guide line. Study arrow heads in Fig. 2.

Place the pencil point in the third hole from the top and move instrument to the left terminal for the guide line.

Place the point in the fourth hole from the top and move the instrument to the right. You have now completed one set of three guide lines. You can make two more sets by continuing the same operations as above. In order to draw more than three sets of lines the instrument and T square must be moved toward the lower edge of the paper. Shift both so that the *extreme upper hole* in the column is directly above the last line you drew before the instrument was shifted. Do not draw a line with a pencil in this upper hole: put the pencil in the *second* hole from the top and draw lines as before. The extreme upper hole is there only to give the proper spacing between lines when the instrument is moved. Study the lines in Figs. 1, 2, 3, 4.

In the use of drawing ink, one other caution should be observed. Although the drawing ink dries rapidly, the width of the line used in making graphs puts upon the paper a considerable amount of ink. Consequently, it takes an appreciable time for wide lines to dry. Care should be taken, therefore, to make certain that the ink is thoroughly dry before proceeding with a second line. Blotting paper cannot be used successfully with drawing ink.

The relative width of the various lines in each drawing should be observed carefully. This variation in width is made in order to bring out the significance of each part of the drawing. Thus, in Fig. 3 the plotted curve is the heaviest, the base line somewhat lighter and the grid lines the lightest.

REPRODUCTION OF GRAPHS

Only a single copy of a graph may be required for many problems. There are times in office work, however, when several copies are desirable; in such cases, it is essential that an easy method of reproduction be available. In case the graph is drawn upon tracing paper or tracing cloth, reproductions may be obtained by the use of blue-print paper. These, however, do not give the finished appearance that some other methods do. Leaving out of consideration the methods used by the printer, graphs can be reproduced conveniently either by a photolithographic process or by photostating. Companies can be found in the large commercial centers which make a business of the photolithographic process of reproduction. When only a few copies are wanted, however, this process is more expensive than photostating. The photostat process is merely a photographic process in which the photograph is taken on sensitized paper. The first copy is always the reverse of the original in color value. That is, if the original drawing is on white paper with black lines, the first photostat copy, commonly called the "negative," will have white lines on a black background. When this is copied, the result will give black lines on a white background as in the original.

If the photostat process is used, it must be remembered that certain colors will not be reproduced. In fact, certain of them may entirely disappear. Thus, blue lines on a white background will not be reproduced in a photostat process. Photostat machines are available in most of the larger cities so that for most business purposes, where only a few copies of a graph are required, the process is not only accurate, but inexpensive.

CHAPTER III

ARITHMETIC SCALE LINE GRAPH

Figures 8 and 9

PURPOSE

As the title suggests, the data are to be plotted on an arithmetic scale, with the plotted points connected by a line. This type of graph is perhaps the most used of all types. In connection with business problems it is most common to find a graph of this type with the horizontal scale representing time and the vertical scale representing quantities which correspond to the various intervals of time. The purpose of such a graph is to show absolute changes in the data over these intervals of time, which may be either years, months, or weeks, or, in rare cases, days. Since the procedure taken in the construction of the grid for this graph also is basic in drawing a number of other graphs, it should be understood thoroughly.

CONSTRUCTION

Arithmetic scales are used for the scale divisions on both axes, because the graph shows the absolute values of the quantities for each given unit on the horizontal scale.

The *field* ordinarily should extend from the zero or base line of the grid. This means that the first grid value on the vertical axis should be the zero value of the vertical scale. Unless this is included no comparison can be made of the relative magnitude of the values of the plotted points since this relative magnitude corresponds to the vertical distance from the zero line. In *special cases* when the lowest value in the series is far from zero, space may be conserved or the change in value from point to point may be magnified by allowing the field or grid to start at zero, cutting it immediately and beginning again at some higher value. This cut should be shown by two jagged lines extending entirely across the grid in order to indicate that a part of the field has been torn out (see Fig. 22).

A graph is most effective if a minimum of grid lines is drawn. Both the horizontal grid lines and the vertical grid lines should be equally spaced. When the lines are *inked,* the zero or base line is made heavier than the other grid lines to indicate that it is the standard. If the data are expressed in percentage form, the 100% line should be emphasized as well as the base line.

Points are plotted in the center of the horizontal spaces and the points are connected by straight lines. When the curve is inked, the plotted line also is made heavier than the grid lines so that it will stand out clearly from the background. When more than one line is plotted, the lines should be distinguished (*a*) by a difference in character, such as solid or dashed, or (*b*) by a difference in color. In addition, the lines should be identified, preferably by a direct label near their left-hand extremity.

In order that the steps in the construction of this graph may be made clear, they now will be indicated in detail. Since the steps for other types of graph are similar to the one outlined, the details of the steps will not be repeated in the chapters which follow.

STEPS IN THE CONSTRUCTION OF THE GRAPH

1. After the paper has been pinned on the drawing board in such a way that the longer lower edge is in line with the upper edge of the T square, draw a vertical line on the right-hand side of the paper ½ inch in from the edge of the paper. This is line *A* in Fig. 3.

2. Next the horizontal base line is drawn at a distance of 1¼ inches to 1½ inches above the bottom edge of the paper. A variation of ¼ inch is allowed to accommodate either (*a*) the height of the graph described in step 3, or (*b*) the amount of information necessary at the bottom of the graph. This line *B* should be 9 inches in length and drawn from line *A*.

3. The height of the grid should be 4 to 5 inches. The exact height will depend upon the scale used, which in turn will depend upon the range of the data. To determine this, find the maximum value in the series of data to be plotted. A little margin should be added so that the plotted line will not come to the top of the grid. Thus, in the data given, page 24, the maximum value is a little more than 350,000. Consequently, 400,000 was selected for the top grid line. If the scale is to be in thousands of units

this would mean 400 *units* to be put in a space of 5 inches. Consequently, the most convenient scale for this graph would be a scale divided into 80 parts to the inch. Since the statistician's rule, described in Chapter II, is not provided with an 80 scale, it will be necessary to reassign values to either the 20 or the 40 scale. If the 20 scale is used, the reassignment of values would be 2 = 80, 5 = 200, and so on, or if the 40 scale is selected, the values would be 1 = 20, 2.5 = 50, and so on. As has been pointed out above, it is not only unnecessary but also undesirable to put in many grid lines (see Hibben Truck Company case, Chapter I). Consequently, every 50 units were marked. It is a question whether the graph might not be equally effective if every 100 units only were marked.

If it is desired to plot the same series of data for the years 1927 to 1930, as shown in Fig. 9, the top grid line is selected to represent 700,000. There is no scale which can be used directly to give the necessary subdivisions in the allotted 5 inches so that the statistician's scale must be used as a diagonal scale, as described on page 10 and shown in Fig. 4.

4. Since the height and length of the grid have been determined, the grid can be closed in.

5. Only a few vertical lines are desirable. If data by months are plotted, the annual division of the years only is necessary. Figure 8 will illustrate this. If data by years are plotted, only sufficient vertical lines to guide the eye should be used. A number of very light, vertical pencil lines may be used to aid in plotting. These will be erased in the final drawing.

6. The pencil guide lines referred to in 5 may be drawn to represent the midpoint of the time interval so that each point may be plotted on a guide line. The vertical lines to be inked finally, however, separate the time intervals, and the plotted points are in the middle of the time intervals. Because there is always confusion when the data are plotted on the dividing line, the data are plotted in the middle of the spaces. The confusion arises, for example, when a number is plotted on the line dividing two months. In such a case the reader is left in ignorance as to whether the plotted point applies, for example, to December of one year or to January of the next year. In other words, the vertical lines indicate divisions between intervals of time so that the plotted points are assigned to the time interval such as a month and not to the dividing line between two months.

7. In lettering, general instructions given above should be followed (see Figs. 5, 6, and 7).

INTERPRETATION

The interpretation of the graph may be made in either of two ways, (*a*) the comparison of two or more individual items, or (*b*) the general picture which a series of items presents.

The objective in comparing two or more items is to picture the difference in their magnitudes. For example, if we had a line graph as shown in the illustration, we could get at once a picture of the production of passenger cars for the maximum month of each of two or three years. This occurs apparently in one of the spring months. On the other hand, a shift of the eye immediately brings a comparison of the production for the Decembers. In case two series, such as passenger cars and trucks, are plotted, the eye can immediately compare the difference in magnitude in the production of cars and of trucks for any particular month which may be desired.

The purpose of the second type of comparison is to secure a general picture. This comparison occurs when an idea of one or more of the four fundamental elements which a time series presents is desired. These are (1) trend, (2) seasonal variation, (3) cyclical fluctuations, (4) accidental changes. Their meaning in connection with statistical series is described fully in Chapter V of Book II.

In making either type of comparison, the important fact of comparing absolute magnitudes should be emphasized. The graph is unsatisfactory when making comparisons on a relative or percentage basis, unless data are in percentage form. A semilogarithmic or ratio grid should be used when percentage comparisons are desired. This form of graph is discussed in the following chapter.

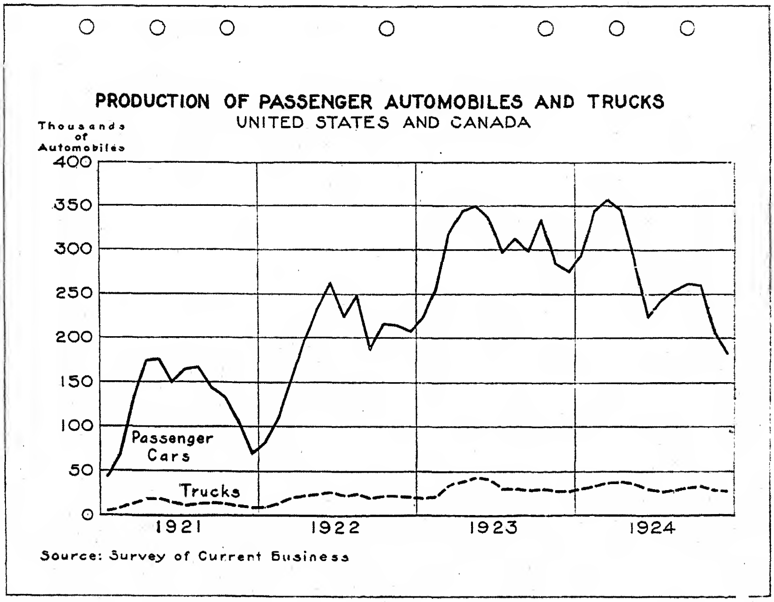

PRODUCTION OF PASSENGER AUTOMOBILES AND TRUCKS
UNITED STATES AND CANADA
Thousands of Automobiles
400
350
300
250
200
150
100
50
0
Passenger Cars
Trucks
1921
1922
1923
1924
Source: Survey of Current Business

FIG. 8.

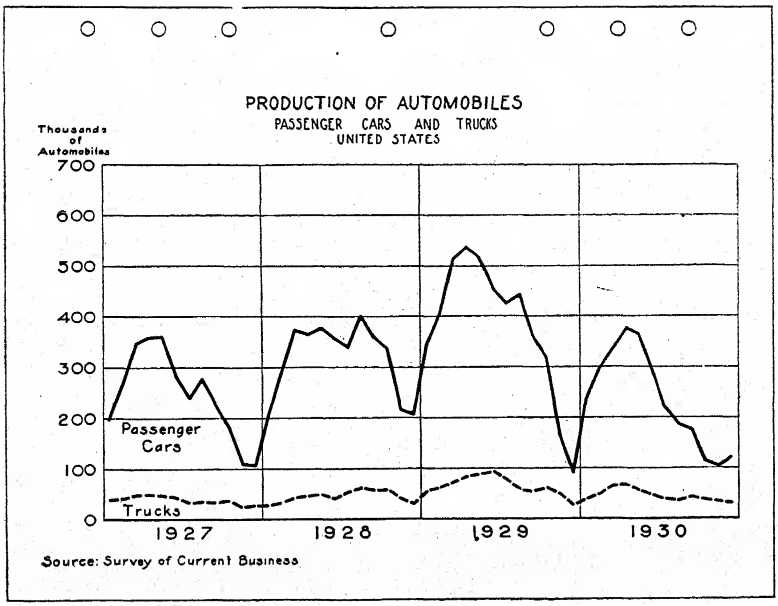

PRODUCTION OF AUTOMOBILES
PASSENGER CARS AND TRUCKS
UNITED STATES
Thousands of Automobiles
700
600
500
400
300
200
100
0
Passenger Cars
Trucks
1927
1928
1929
1930
Source: Survey of Current Business

FIG. 9.

Data to Accompany Arithmetic Line Graph and Logarithmic Line Graph[1]

The following extract from tables published in the *Survey of Current Business* shows the number of passenger automobiles produced during each month for the years shown and also the parallel production of automobile trucks.

Production of Passenger Cars, U. S. and Canada

Month	1921	1922	1923	1924
January	43,086	81,696	223,822	293,798
February	68,088	109,171	254,782	343,431
March	130,263	152,962	319,789	356,976
April	176,439	197,224	344,661	346,320
May	177,438	232,462	350,460	286,146
June	150,263	263,053	337,442	224,965
July	165,616	225,103	297,413	244,387
August	167,756	249,408	314,431	255,073
September	144,670	187,711	298,964	263,411
October	134,774	217,582	335,041	260,839
November	106,081	215,362	284,939	204,313
December	70,727	208,016	275,472	182,023

Production of Trucks, U. S. and Canada

Month	1921	1922	1923	1924
January	4,831	9,596	19,739	30,627
February	7,830	13,360	22,178	32,756
March	13,328	20,036	35,298	36,270
April	18,070	22,665	38,102	37,766
May	18,070	24,120	43,757	35,112
June	14,328	26,354	41,176	28,884
July	11,136	22,083	30,708	26,227
August	13,400	24,711	30,884	28,503
September	13,978	19,497	28,592	31,829
October	13,149	21,830	30,153	32,332
November	10,487	21,972	28,085	27,766
December	8,656	20,406	27,772	27,324

[1] For graph see page 33.

Production of Passenger Cars, United States

Month	1927	1928	1929	1930
January	199,650	205,646	345,545	236,145
February	264,171	291,151	404,063	296,461
March	346,031	371,821	511,577	335,720
April	358,682	364,877	535,878	374,913
May	358,725	375,863	514,863	362,522
June	280,620	356,622	451,371	289,245
July	237,811	338,792	424,944	222,459
August	275,585	400,593	440,780	187,037
September	226,443	358,872	363,471	175,311
October	184,042	339,976	318,462	115,476
November	109,758	217,256	167,846	102,358
December	106,083	205,144	91,011	122,645

Production of Trucks, United States

Month	1927	1928	1929	1930
January	39,258	26,082	53,428	38,657
February	40,564	32,645	60,247	49,457
March	48,482	41,506	71,799	64,204
April	47,700	45,227	84,346	67,560
May	46,923	49,920	88,510	54,370
June	43,197	40,174	93,183	45,771
July	31,585	53,284	74,842	39,663
August	34,409	60,705	56,808	35,758
September	33,867	56,422	51,576	41,157
October	36,640	57,136	60,687	38,343
November	24,612	39,679	48,081	32,785
December	27,488	27,991	27,513	31,531

CHAPTER IV

LOGARITHMIC SCALE LINE GRAPH

Figure 12

PURPOSE

The title of the chapter indicates that in this case the data are to be plotted on a logarithmic scale. Two other common names which are given to this type of graph are the semilogarithmic graph and the ratio graph. The purpose is to exhibit ratio or percentage relationships instead of absolute changes as in the arithmetic scale line graph.

One of the common but simple types of analysis in business problems makes use of the ratio of a given figure to another figure; thus, a business executive often speaks of the percentage gain of his sales for one year over those of the preceding year, or compares the volume of his business with the total for all other similar businesses by means of a percentage figure. As another illustration, the business man often thinks of the growth of his business from year to year in terms of percentage; that is, he likes to know whether on the average it grows, say 10% a year. These are common illustrations of what may be called a "ratio analysis."

With the idea of ratio or relative size in mind, we should like to have some kind of a grid to show graphically whether the ratio or the percentage that we are talking about remains the same, or changes over a period of time. A simple example will present the ideas fundamental in the construction of a grid showing a constant ratio or constant percentage change. If we have the series 2, 4, 8, 16, each number is twice the preceding one. This is the same as saying that each number is 200% of the number immediately preceding it. If, then, a grid is drawn in such a fashion that the horizontal line representing 8 is just as far above the horizontal line representing 4 as 4 is above 2, we shall obtain three horizontal lines equidistant from one another. Each of the equal spaces will represent the fact that the line at the top of the space is in value just 100% *more* than the value of the preced-

ing line at the bottom of each space. Thus, we obtain three lines on a grid which represent a constant percentage change by a uniform distance. Similarly, the line corresponding to 16 can

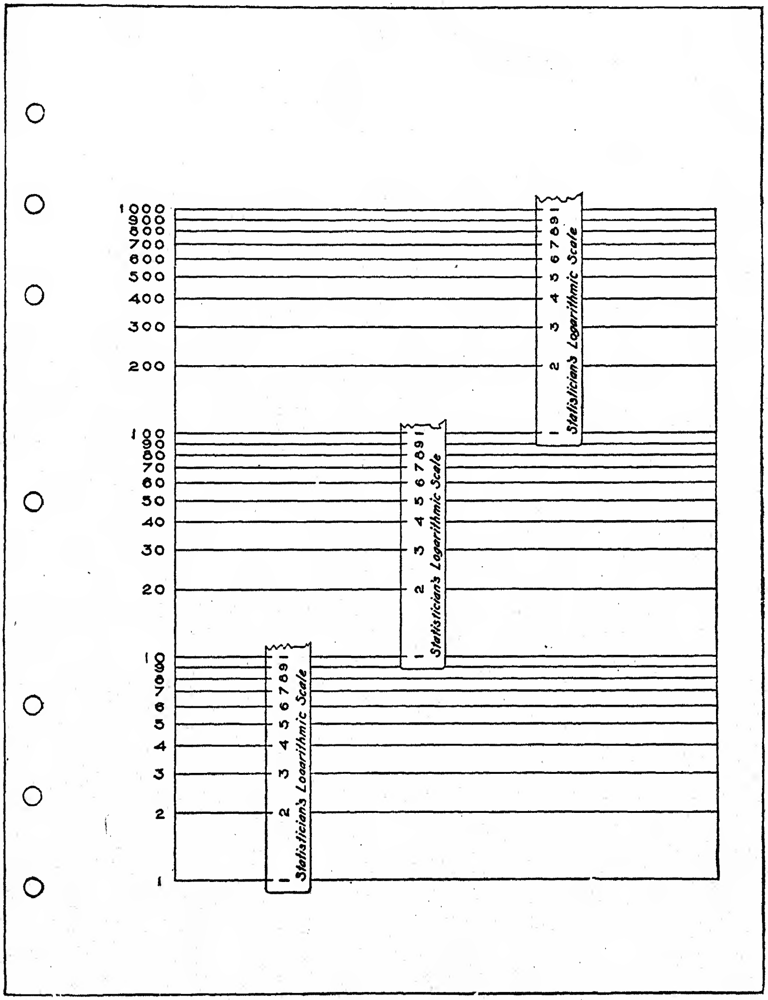


FIG. 10.

be located. The positions of the intermediate lines which we should like to use in plotting are located by a more complicated process. For our purposes it is sufficient to know that they may be determined by the use of logarithms. It is unnecessary

to understand the details of the principles of logarithms in order either to make or to use ratio graphs.

To illustrate an important consequence of the ratio scale, let us examine the numbers 1, 10, 100, and 1,000. Each number obviously is 10 times the preceding number. If the ratio principle is to be used in constructing the grid, then the vertical distance between 1 and 10 must be equal to that between 10 and 100, which in turn is the same as that between 100 and 1,000. Thus, within a vertical distance three times that necessary for plotting numbers between 1 and 10, we can plot all numbers between 1 and 1,000 (see Fig. 10). In contrast to this, an arithmetic grid would require a vertical distance 100 times that considered necessary for plotting numbers between 1 and 10 to enable the statistician to plot numbers between 1 and 1,000. The direct result of this property is that we are enabled to compare fairly small numbers with large ones, such as those of a small business with the corresponding figures representing the total for an industry.

It is common practice to refer to a division of the scale from 1 to 10 units as a "bank," "tier," or "cycle." A division from 10 to 100 would be a second bank and so on. Thus, corresponding grid lines in successive banks are 10 times those in each preceding bank.

This type of grid should *not* be used when it is desired to picture absolute changes in a series of data.

CONSTRUCTION

The first step in constructing a grid on the logarithmic scale is to draw the lowest horizontal line. There can be no base line for a logarithmic grid because the zero line on a logarithmic scale would be an infinite distance below any line which may be drawn. This first horizontal line, therefore, is not numbered zero for a base line but is numbered to correspond to a value slightly less than the lowest value to be plotted. Next, the height of the grid must be determined. The data are examined to find the extreme high and low values in order to determine the appropriate number of banks which are to be drawn. Knowing the range of the data and hence the number of banks, the draftsman next selects an appropriate scale. On the statistician's scale there are three different sized logarithmic scales. These scales do not differ in kind but simply differ in the dimension required for a single

bank. If two banks are required, the selected scale will give the first of the banks. The statistician's scale is then moved up so that the same scale points are repeated a second time (see Fig. 10). Occasionally it will be found that no one of the three scales will be entirely satisfactory for the data in hand. In such a case, the scale must be so placed that an enlarged or reduced scale of the dimensions required may be constructed in the manner described on page 10, Chapter II, and as illustrated in Fig. 4, page 12.

It is unnecessary to begin every bank with unity. If the lowest number happens to be 36, for example, the lowest horizontal line on the grid can be located from the number 3 on the statistician's logarithmic scale. The last point marked by the first application of the selected logarithmic scale consequently will be at 10 on the scale. Then the scale is repeated. If, for example, the highest number is 52, the point 6 or 7 of the scale on this second application might be used as the uppermost line.

Just as in the case of the arithmetic scale line graph, only a few horizontal and vertical lines should be ruled in. Usually the horizontal lines correspond to whole numbers on the logarithmic scale. Because the distance between 1 and 2 on a logarithmic scale is disproportionately large, sometimes it is wise to put in the line corresponding to 1.5. Since the lowest horizontal line does not represent a base line, it should not be inked more heavily than the other lines.

INTERPRETATION

To some extent the interpretation of this graph has been included in the discussion of its purpose. Briefly stated, the interpretation may take either of two directions, (*a*) it may enable us to measure the rate of change in a single series of data, and (*b*) it may enable us to make a comparison of the rate of change of two or more plotted lines.

It is possible to recognize changes in the rate of growth because the grid is so constructed that equal vertical distances always represent the same proportion or the same rate of change. For this reason, a series of data which represents the same percentage rate of increase, such as the compounding of a sum of money in a savings bank, is a straight line when plotted on this type of grid. If several points on the plotted line appear to be in the same

general direction, the percentage rate of gain is nearly the same. If the general direction of the plotted points changes, it means that the percentage rate has changed also. It should be noted that a series of points, which plot as a straight line on an arithmetic

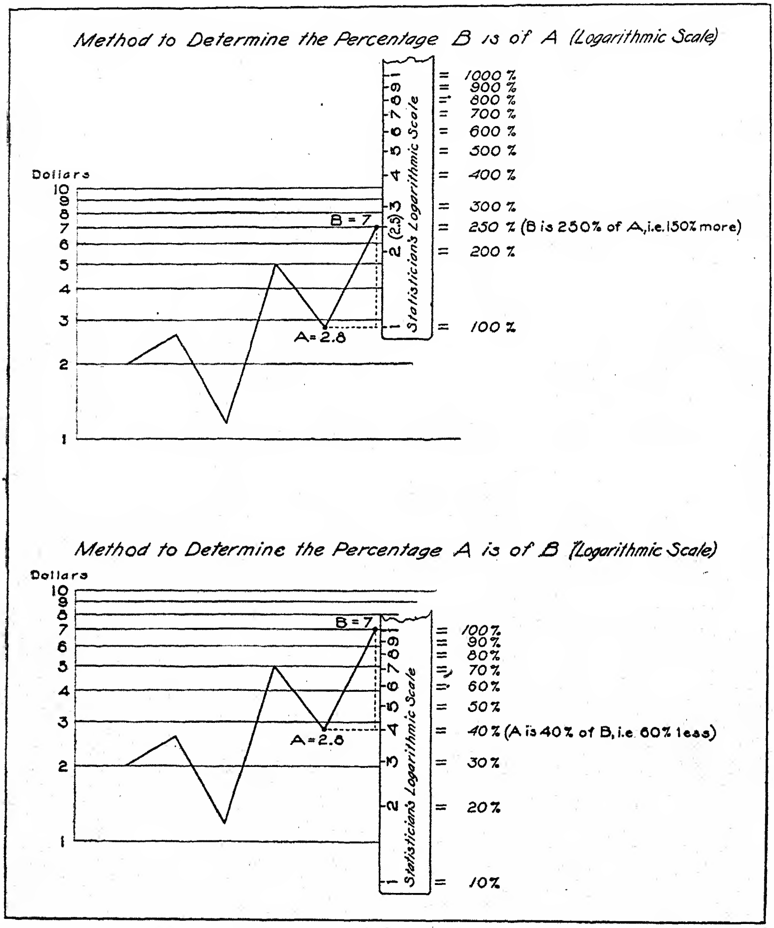


FIG. 11.

grid and which represent equal absolute changes, plot as a curved line, concave downward, on a semilogarithmic grid.

One of the simplest and yet most valuable uses to which a semilogarithmic graph can be put is in estimating percentage

change. This can be done by a purely graphical method. Let us suppose two points, A, 2.8, and B, 7, are plotted on a semi-logarithmic grid and let us suppose that it is desired to know what percentage B is of A (see Fig. 11). The first step is to determine graphically the difference in height of B over A. This may be accomplished by drawing a horizontal line from A until it passes under B. Then the distance of B above this horizontal line may be read off from a statistician's logarithmic scale which is, of course, the same as that used in constructing the grid. The 1 on the scale should be placed on the lower mark corresponding to A and the value on the scale read off opposite B. In this case, the 1 on the statistician's scale is interpreted to mean 100%. The value corresponding to the upper mark, or B, is then read from the statistician's scale. The scale reading corresponding to B is 2.5. This means that the value B is 250% of A.

On the other hand, if we desire to find what percentage A is of B, then B will represent the base or 100% value. In this case we place the 1 at the upper end of the scale on B, and read off the value opposite A. For example, in the figure, A is found to be 40% of B. By subtraction of this percentage from 100, the percentage decrease can be found if desired.

In case the logarithmic scale is either a contraction or an expansion of one of those on the statistician's scale, a strip showing the scale division transferred from the edge of the grid will be found more convenient than the statistician's scale. This will be necessary also whenever one of the commercially printed grids is used.

Although the figure is drawn in such a fashion that B is higher than A, B might be lower than A. In this case B would be a percentage of A, which is less than 100%. Obviously this corresponds to the second case explained above.

The second type of interpretation is found in connection with a comparison of two or more plotted lines. If, in general, the lines seem to run at about the same distance apart when plotted, it means that the percentage changes from time to time are practically the same in one series as in the other. This is a very important characteristic because it enables the comparison of two series where the size of the numbers differs radically. For example, it enables a person to compare the growth of a particular business with the growth of all similar establishments whether large or small. If these two lines are put on an arithmetic

grid, the *total of all similar businesses* may appear to outstrip the growth of the single business so far that the small concern may appear to be practically stationary, whereas actually the march of its progress over the course of time may be as significant as that for all similar businesses or even more so (see Figs. 9 and 12).

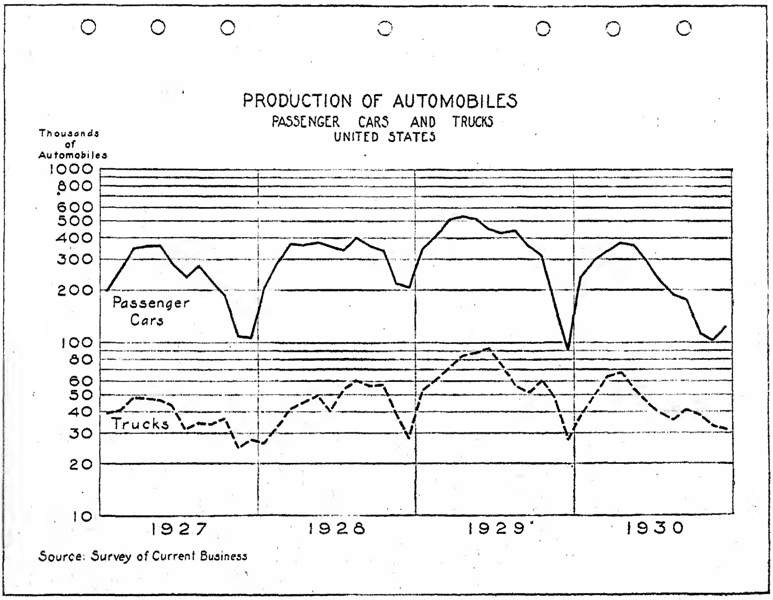

PRODUCTION OF AUTOMOBILES
PASSENGER CARS AND TRUCKS
UNITED STATES
Thousands of Automobiles
1000
800
600
500
400
300
200
100
80
60
50
40
30
20
10
Passenger Cars
Trucks
1927
1928
1929
1930
Source: Survey of Current Business

FIG. 12.

CHAPTER V

BAR GRAPH

Figure 13

PURPOSE

The bar graph is used ordinarily to present a comparison of the absolute magnitude or size of the quantities by means of a series of "bars." The length of the bars measures the quantities from the vertical axis according to a common arithmetic scale while the width of the bars usually remains constant.

The data presented by bar graphs are of two sorts, one in which there is no time element, and the other in which the time element is present. When the time element is lacking the bars usually are arranged in order of size, since the purpose is to compare the size of the quantities. When the series vary with time, two arrangements are possible. In one case the bars are grouped so that they show the same series at different periods of time; in the other case the bars are arranged so that they show magnitudes of different series at the same time.. For example, if we have the sales of Ford cars for the years 1915, 1920, and 1925, and the total number of cars including Fords sold in the United States for these same years, we can group the bars in either of two ways. The first is to arrange in sequence the sales of Ford cars. This will result in a group of three bars followed by a second group of three bars which will show the sales of all cars. Here the change in the sales of Ford cars over a period of years is of fundamental importance and is the fact to be emphasized by the graph. On the other hand, the bars might be arranged so that for 1915 the bar representing the sales of Ford cars would be adjacent to that represent ng the sales for all cars. With such an arrangement the comparison of Ford cars with the sales of all cars would be of primary importance, and the change in these values over a period of time would be secondary.

A special form of the bar graph is that which exhibits data in percentage form. Bars representing percentages of the total or

percentage changes from a preceding period are used. In the first type each of the bars represents a significant percentage of the total. The arrangement of a group of such bars is ordinarily according to size. In the second type negative changes are portrayed by bars extending to the left of the axis, while positive changes are shown by bars to the right. This type of graph is used by the Federal Reserve Bank of Boston in its *Monthly Review* to compare percentage changes in check payments from those of a preceding period in typical cities in the Reserve District.

CONSTRUCTION

The base or zero line in the bar graph is the vertical axis. It is made heavier than other grid lines in order to emphasize the fact that it is the base or starting point in the measurement of the absolute values. Since the purpose is to compare absolute magnitudes, an arithmetic scale is used. The scale numbers are located just below the grid. Values are measured horizontally and to the right of the base or zero line, except in the case of negative values (see Fig. 13). The scale caption is centered below the scale numbers; both are printed horizontally. The grid guide lines do not pass through the bars but stop at their edges.

The exact width of the bars will depend upon the space available and the number of bars presented. It is customary to separate individual bars by a space equal to from one-half to two-thirds of the width of a single bar. Where bars are arranged in groups, an additional space is required between groups, usually the full width of a bar. The outline of the bars should be drawn in ink and the enclosed areas differentiated by colors or shadings.

Numerical data should never be placed at the variable end of the bars since this creates confusion and is likely to cause the reader to add unconsciously the numerical data to the length of bar and thus misinterpret the relationships. The numerical value which each bar represents need not be printed on the graph. If, however, the numerical value is to be stated for each bar, it should be shown at the base of the respective bars and outside the grid.

The horizontal bar graph is particularly advantageous where the identification of each bar requires a considerable space. As noted above and as shown in Fig. 13, the graph makes possible the use of a considerable amount of lettering to identify each bar.

If occasion demands, a bar graph may be turned so that the bars become columns. Consequent changes in labeling and scales, which will be necessary, are obvious.

INTERPRETATION

To interpret a bar graph properly, the length of one bar is compared with the length of another bar either on a quantity basis or, in the case of a percentage arrangement, in terms of proportion. The graph presents the relationship of two or more numerical values.

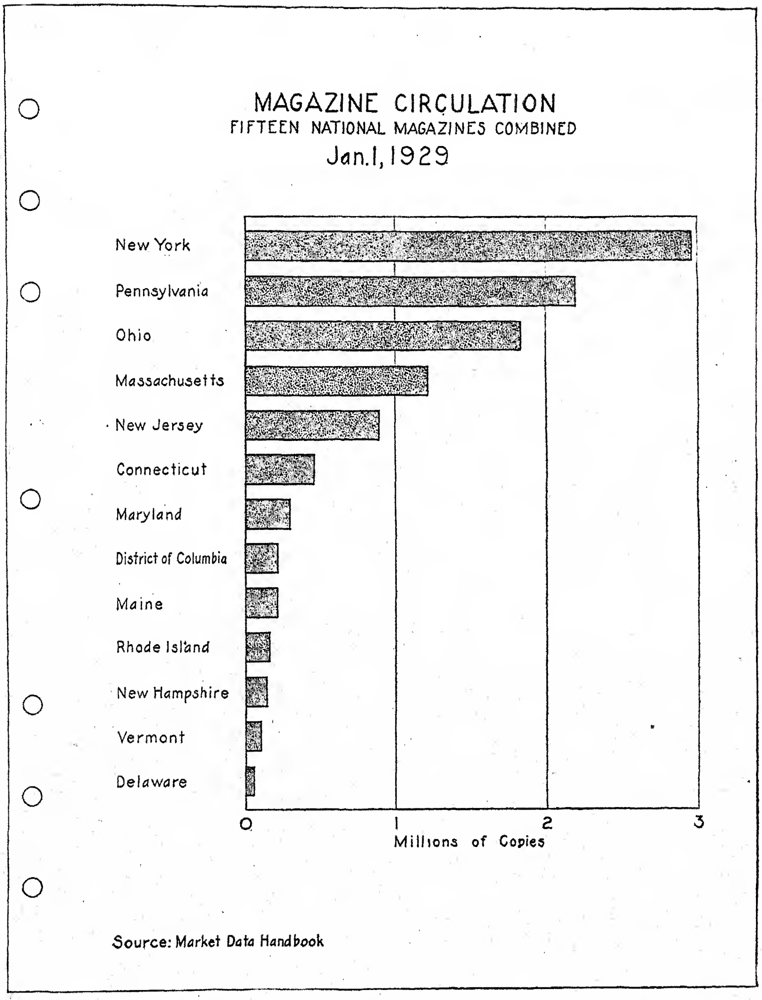

MAGAZINE CIRCULATION
FIFTEEN NATIONAL MAGAZINES COMBINED
Jan. 1, 1929
New York
Pennsylvania
Ohio
Massachusetts
New Jersey
Connecticut
Maryland
District of Columbia
Maine
Rhode Island
New Hampshire
Vermont
Delaware
0
1
2
3
Millions of Copies
Source: Market Data Handbook

FIG. 13.

Data to Accompany Bar Graph

The following information is from the *Market Data Handbook* published by the U. S. Department of Commerce, Table 1, and represents the circulation of fifteen national magazines combined as of January 1, 1929.

State	Number of Copies
New York	2,959,875
Pennsylvania	2,204,998
Ohio	1,845,037
Massachusetts	1,222,863
New Jersey	893,142
Connecticut	462,593
Maryland	303,603
District of Columbia	222,889
Maine	219,317
Rhode Island	167,088
New Hampshire	148,818
Vermont	108,169
Delaware	56,738

CHAPTER VI

COMPONENT COLUMN GRAPH

Figures 14 and 15

PURPOSE

A component column graph is designed to present comparisons of the absolute magnitudes of quantities or, if constructed on a percentage basis, the relative size of parts to a total. This graph is similar to the bar graph in purpose, but differs from the bar graph in that the horizontal axis is used as a base so that instead of horizontal bars we have vertical columns. A component column graph carries the idea of the bar graph one step further, because the columns are divided into segments which represent the component parts of the whole. Thus, in the component column graph, each column represents a group of several facts at a given time. For example, in one of the columns, the height of the column may represent the total sales of a given company. This may be broken down into sales by several lines of goods so that the column representing total sales is broken up into segments, each representing the amount of sales of one line.

Two types of data may be used, depending on whether different totals and their components as of a given time or the values of the total and components of one series at different times are included. The first type of data may be illustrated by total expense and individual expense items for several department stores in a given year, and the second type by total freight car loadings and by classes over a period of years as shown in the accompanying graph. For data of the second type, when the emphasis is upon changes in the series and its components over a period of time, the belt graph, which is discussed in Chapter XIII, may be used.

CONSTRUCTION

An arithmetic scale is used in the construction of the component column graph. Use of the semilogarithmic scale is not

good practice because the base line cannot be represented on a grid. When ratios are to be shown, a semilogarithmic line graph should be used. The scale should be in terms of absolute units when the data to be plotted are expressed in the original units. A percentage scale should be used when the series are in the form of percentages. In this case, the total figure for each column is equivalent to 100%. This forms a special case of the component column graph, in which the heights of all the columns will be the same. An example is shown in Fig. 15.

The width, grouping, grid lines, and spacing of columns follow the practice used in the construction of the bar graph, except that vertical columns rather than horizontal bars are used.

The columns are constructed by cumulating the values of the components. It is customary to plot the value of the largest component of the series from the base line and to measure the value of the next largest component from the top of the one already plotted and so on. A miscellaneous item usually is placed at the top, regardless of size. The same sequence of components is maintained for each of the columns. The point corresponding to the top of the last component plotted should be checked to make certain that its distance from the base line is equivalent to the total value represented by the column. After the outline of the column is drawn, each segment is indicated by a horizontal line across the column at the appropriate distance from the base. An example of this type of graph is shown in Fig. 14.

Corresponding segments of the different columns should be differentiated by the use of like colors or cross-hatching. A key below the grid should be used to indicate the meaning of the device used for differentiating segments.

INTERPRETATION

Comparison of the heights of columns and of components gives comparison of absolute magnitudes, or of percentages in the case of the 100% component column graph. Because the components which are located at the bottom of the columns are measured from a common horizontal level, a clear idea of their relationship is obtained. When the other components are compared, however, each may be upon a different level, and it is somewhat difficult to secure an accurate impression of their true proportions.

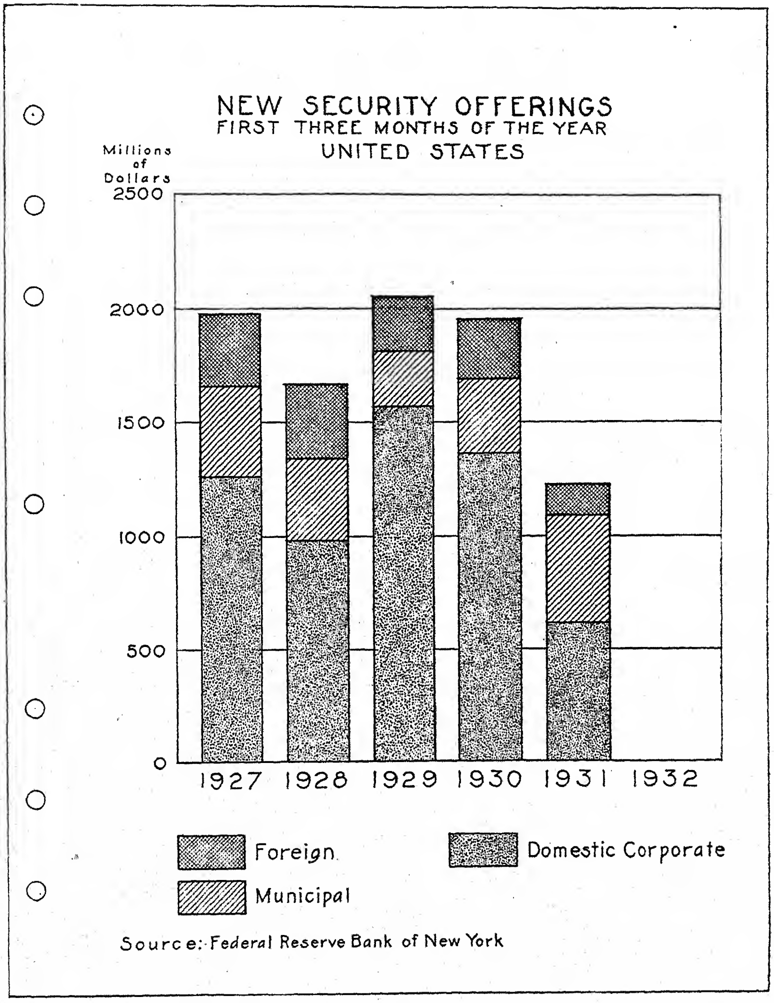

NEW SECURITY OFFERINGS
FIRST THREE MONTHS OF THE YEAR
UNITED STATES
Millions of Dollars
2500
2000
1500
1000
500
0
1927 1928 1929 1930 1931 1932
Foreign
Municipal
Domestic Corporate
Source: Federal Reserve Bank of New York

FIG. 14.

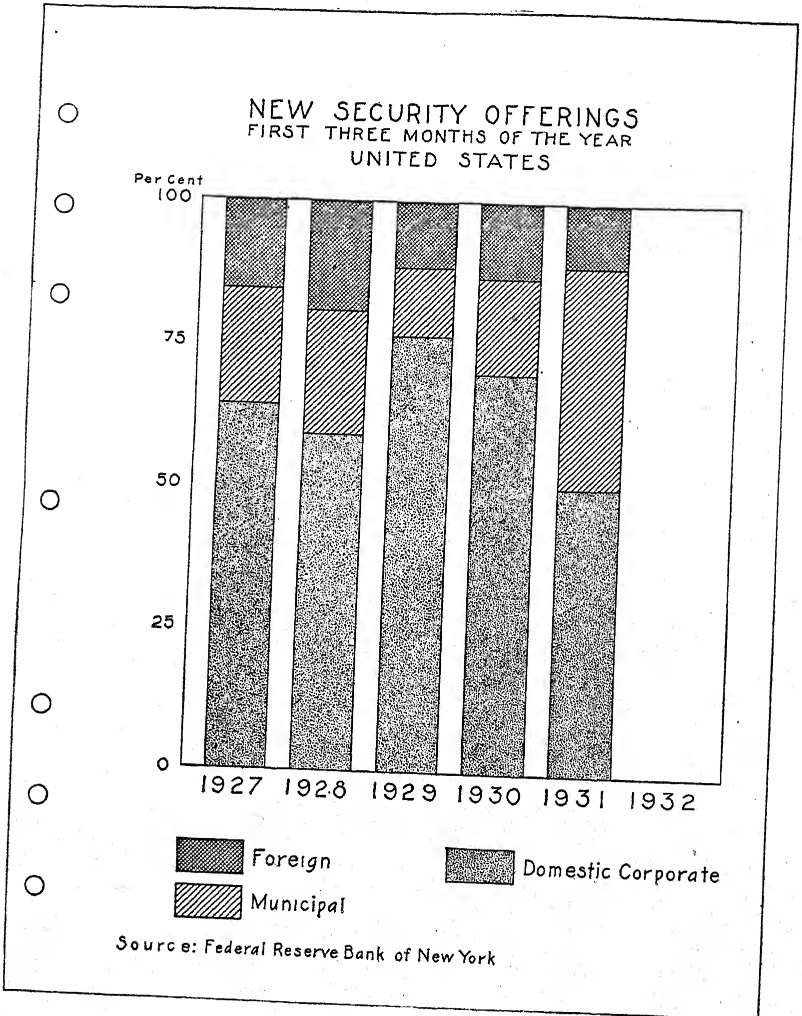

NEW SECURITY OFFERINGS
FIRST THREE MONTHS OF THE YEAR
UNITED STATES
Per Cent
100
75
50
25
0
1927 1928 1929 1930 1931 1932
Foreign
Municipal
Domestic Corporate
Source: Federal Reserve Bank of New York

FIG. 15.

Data to Accompany Component Column Graph

The following information is from page 37 of the May 1, 1931 issue of the *Monthly Review* of the Federal Reserve Bank of New York. It shows new security offerings classified by domestic corporate issues, municipal issues, and foreign issues as well as the total figures for the first three months of each year. The lower portion of the table expresses each class in terms of per cent of the total.

Year	Total	Domestic Corporate	Municipal	Foreign
		(Millions of Dollars)		
1927	1,978	1,267	399	312
1928	1,668	983	360	325
1929	2,059	1,570	246	243
1930	1,952	1,364	326	262
1931	1,228	614	474	139
		(Per Cent)		
1927	100	64.06	20.17	15.77
1928	100	58.93	21.59	19.48
1929	100	76.26	11.94	11.80
1930	100	69.88	16.70	13.42
1931	100	50.04	38.63	11.33

CHAPTER VII

FREQUENCY GRAPH

Figure 27

PURPOSE

The purpose of the graph is to picture a frequency distribution and to show the varying number of items of a series of data which may occur within certain class groups. For example, in the game of bridge we might determine from a succession of hands the number of times that no aces, one ace, two aces, three aces, and four aces occur. Obviously here there are five classes, and the number of times which any one of the five conditions occurs would be the frequency.

CONSTRUCTION

The principle of construction is the same as for a column graph with the exception that there is no space interval between the columns.

In regard to numbering the limits of the class intervals on the horizontal scale, the columns may not be wide enough to allow space for the first as well as the last numbers in each class interval in a single horizontal line. In such a case, it is customary to print these scale (class interval) numbers under the columns in two lines or rows, the upper row being the first numbers in each class and the lower row, staggered slightly to the right, representing the last numbers in each class. The numbers in the lower row are preceded usually by a dash or the word "to" to indicate that the numbers given include the extreme limits. Figure 27 illustrates this type of graph. It also shows that the suggestions just made in regard to scales cannot always be followed.

The vertical scale is an arithmetic scale to show the number of items in each class interval.

In some cases it is desirable to connect by means of a straight line the midpoint of the top of each column with the adjacent column. It will be noticed that where there are no columns

the straight line is drawn from the adjacent column to the base line. This occurs always at the beginning and at the end of the distribution and may occur at some point within the distribution for a class in which no items are reported. These points are shown clearly in Fig. 27. The figure formed by connecting the tops of the columns when considered alone is referred to as a frequency polygon.

INTERPRETATION

The significance and interpretation of the frequency distribution are described in detail in Chapter II of Book II.

CHAPTER VIII

CUMULATIVE GRAPH

Figure 16

PURPOSE

The objective of the cumulative graph is to present cumulative or progressive totals at successive intervals. It simply represents at each point the total of all the preceding separate items. Since businesses consider the year a major division of time, the progressive totals start with zero at the beginning of the year and increase in size so that the value for December represents the total sales or production for the year. The purpose and principles of the graph are the same if the year is the fiscal year or if the data are weekly or other totals.

CONSTRUCTION

The grid may be either on the arithmetic scale or on the semilogarithmic scale. On an arithmetic scale the plotted line for the cumulative curve always begins at zero. If the semilogarithmic grid is used, however, it is impossible to plot the zero point. The first points to be plotted, therefore, are those representing total sales or production by the end of January if monthly data are used. The progressive values for each month are plotted on the boundary lines since the sales or production are not completed until the end of the month. A point plotted on the boundary line between March and April would indicate that the total sales at the end of March or at the beginning of April were represented by the plotted value.

INTERPRETATION

After a series of values representing the progressive monthly totals has been plotted, the significance of the increasing or decreasing size of the successive increments is disclosed by the shape of the curve. A cumulative curve is especially valuable

when comparing actual sales with planned sales during the year, actual expenses with budgeted expenses, or the actual volume of incoming orders with estimates previously made. These figures show cumulative totals of business to date at each successive interval of time, and in addition show the increments from each interval to the next in striking comparison with the trend of movement from the beginning of the curve. Another use of the graph is the comparison of the total figures to date in any year with those for the previous year. Of course, the progressive monthly totals for the previous year can be plotted so that the graph will present one line for the previous year together with one for the current year.

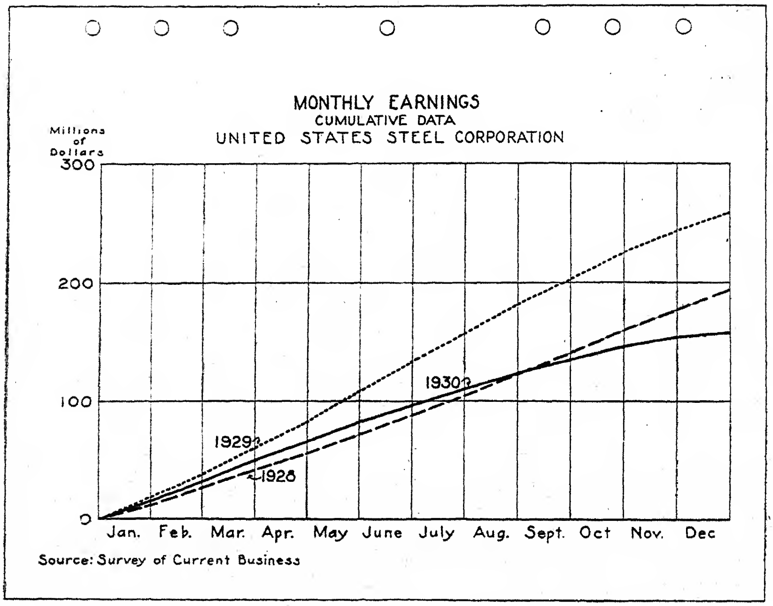

MONTHLY EARNINGS
CUMULATIVE DATA
UNITED STATES STEEL CORPORATION
Millions of Dollars
300
200
100
0
1929
1928
1930
Jan. Feb. Mar. Apr. May June July Aug. Sept. Oct Nov. Dec
Source: Survey of Current Business

FIG. 16.

Data to Accompany Cumulative Graph

The following figures represent the monthly earnings of the United States Steel Corporation as reported in the *Survey of Current Business*. The cumulated figures have been added to show the method pursued in making cumulative graphs.

(Thousands of Dollars)

Date	Monthly Earnings	Cumulative Earnings
1928		
January	11,900	11,900
February	13,581	25,481
March	15,453	40,934
April	13,927	54,861
May	16,647	71,508
June	16,359	87,867
July	16,134	104,001
August	18,597	122,598
September	17,418	140,016
October	19,399	159,415
November	17,365	176,780
December	16,423	193,203
1929		
January	18,759	18,759
February	19,081	37,840
March	22,265	60,105
April	22,561	82,666
May	25,605	108,271
June	24,029	132,300
July	24,303	156,603
August	24,687	181,290
September	21,184	202,474
October	22,066	224,540
November	18,367	242,907
December	15,952	258,859
1930		
January	15,404	15,404
February	16,108	31,512
March	18,104	49,616
April	16,114	65,730
May	16,571	82,301
June	14,377	96,678
July	13,480	110,158
August	13,000	123,000
September	11,515	134,673
October	10,943	145,616
November	7,949	153,565
December	4,191	157,756

CHAPTER IX

MULTIPLE SCALE GRAPH

Figure 17

PURPOSE

The purpose of the multiple scale graph is to present the changes which take place over an interval of time in two or more series of data whose units are widely different in magnitude or different in kind. Since the purpose is to picture the changes which take place over an interval of time, and since the changes to be observed are relative, not absolute, in magnitude, a semilogarithmic grid is used. The graph may, therefore, be a composite of several semilogarithmic graphs.

Although arithmetic scales sometimes are used for multiple scale graphs, their use in most cases is undesirable so that they are not recommended for this particular purpose.

CONSTRUCTION

As in the case of a single scale semilogarithmic graph, which is described in Chapter IV, comparatively few horizontal grid lines are needed. Usually no confusion arises in planning the graph if each new scale is to be a multiple of 10 of the scale already used. The reason for this is that such a multiple scale graph really superimposes each new bank (or tier) of a semilogarithmic grid above the one already drawn. At times it is convenient to use some other constant than 10 as a multiplier in order to set the new scale values. An example of this sort is shown on the accompanying graph. If the spindle hours scale on the extreme left is taken as the base scale, the scale used for spindles causes little difficulty, since it is ten times the original scale. On the other hand, the scale for the capacity differs decidedly. If 50 is the rating set for the lowest line on the capacity scale, then the second numbered line from the top will be rated at 100, because this line represents just twice the value of the lowest line on the original scale. Since the top line represents three times the quantity

of the lowest line, the price index scale for this line should be rated at 150. In other words, the scales must always be proportional. If scales are used which are not proportional to each other, the graph will be unreliable because the plotted lines then will show the data in a distorted and misleading manner.

As in the case of other line graphs, the number of plotted lines on a multiple scale graph should be kept at a minimum. Since the purpose is to compare lines of different magnitudes, or those which represent unlike kinds of units, or both, three lines should be considered a safe maximum in applying a working rule.

INTERPRETATION

Little need be said in regard to the interpretation of this kind of graph, since the objective in making comparisons is exactly the same as that already described in the case of the semilogarithmic graph discussed in Chapter IV.

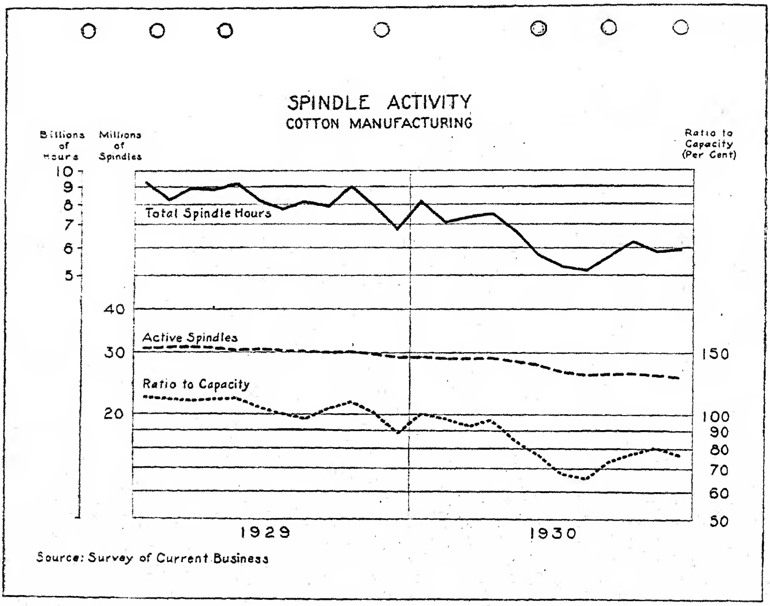

SPINDLE ACTIVITY
COTTON MANUFACTURING
Billions of Hours
Millions of Spindles
Ratio to Capacity (Per Cent)
10
9
8
7
6
5
40
30
20
150
100
90
80
70
60
50
Total Spindle Hours
Active Spindles
Ratio to Capacity
1929
1930
Source: Survey of Current Business

FIG. 17.

Data to Accompany Multiple Logarithmic Scale Graph

Three series from the *Survey of Current Business* are printed below. Plotted as a multiple scale graph, they show certain significant relationships.

Spindle Activity in Cotton Manufacturing

Date	Active Spindles, Thousands	Total Spindle Hours, Millions of Hours	Ratio to Capacity, Per Cent
1929			
January	30,758	9,225	111.6
February	31,008	8,221	110.7
March	31,104	8,910	109.3
April	30,924	8,861	110.3
May	30,397	9,164	110.9
June	30,632	8,160	104.8
July	30,397	7,757	100.3
August	30,237	8,130	97.7
September	30,038	7,881	104.0
October	30,135	9,004	108.7
November	29,649	7,812	100.9
December	29,070	6,770	88.2
1930			
January	29,198	8,173	100.3
February	28,927	7,091	97.7
March	28,898	7,350	92.8
April	28,860	7,503	96.3
May	28,374	6,729	83.6
June	27,642	5,779	76.3
July	26,458	5,301	67.2
August	25,874	5,134	65.2
September	26,087	5,663	73.4
October	26,154	6,239	77.1
November	25,858	5,832	80.1
December	25,526	5,916	76.1

CHAPTER X

STOCK MARKET GRAPH

Figure 18

PURPOSE

This type of graph is used commonly in financial publications to picture the course of the stock market. Since investors desire to know the range of the market price at any particular time, the high and low points are indicated by the location of the top and bottom of a series of solid columns. In some cases a line also is plotted on the graph to indicate such things as the number of shares sold in a given period. The graph is complicated sometimes by the necessity of using several scales (see Chapter IX on the Multiple Scale Graph).

CONSTRUCTION

Since for an extended space of time a larger size of paper than 8½ by 11 inches might be advisable, it is necessary that the proportions of the graph be different from those indicated in Fig. 18.

The vertical columns or lines which indicate the spread in price for the day, week, month, or year are best made by drawing vertical lines with a ruling pen which is set at a proper width. These heavy vertical lines simply connect the high and the low points for the unit of time used on the graph.

An arithmetic or logarithmic scale may be used for this type of graph. Naturally the interpretation will vary with the grid chosen.

INTERPRETATION

The interpretation of the graph includes not only an understanding of the general movements of the stock prices but also the actual spread of prices for the particular unit of time used, whether day, week, month, or year. During active markets this spread of prices is of especial significance in indicating certain phases of the activity of the market. If the number of shares traded is put upon the graph, the activity as represented by volume of business done is included. The volume of business in relation to the prices and the spread of prices is also of material interest.

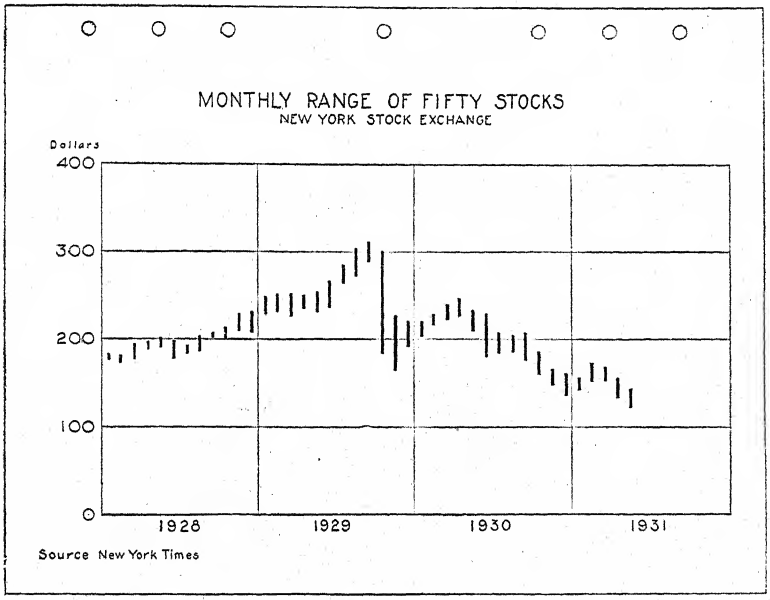

MONTHLY RANGE OF FIFTY STOCKS
NEW YORK STOCK EXCHANGE
Dollars
400
300
200
100
0
1928
1929
1930
1931
Source New York Times

FIG. 18.

Data to Accompany Stock Market Graph

The series below show the high and low values for an index of fifty stocks listed on the New York Stock Exchange by months for 1928, 1929, 1930, and a portion of 1931, compiled by the New York Times Company and published currently in the *New York Times*.

Monthly Range of 50 Stocks

Date	High	Low	Date	High	Low
1928			1930		
January	183.2	176.5	January	220.2	203.0
February	180.8	173.1	February	228.4	215.8
March	194.5	176.1	March	240.2	222.6
April	196.8	188.5	April	245.6	226.4
May	201.1	190.6	May	233.9	209.8
June	198.1	177.8	June	229.2	179.1
July	192.3	183.7	July	207.1	183.4
August	203.6	186.5	August	204.3	185.0
September	207.6	201.3	September	207.8	175.6
October	213.7	200.5	October	186.1	160.0
November	229.5	209.1	November	165.7	147.9
December	231.5	207.1	December	161.9	135.4
1929			1931		
January	248.9	228.4	January	156.6	142.8
February	251.5	231.6	February	173.1	152.0
March	252.1	226.8	March	169.0	153.6
April	249.9	234.9	April	155.8	133.2
May	254.0	230.4	May	143.5*	122.0
June	265.5	235.2	....	....	....
July	285.1	263.7	....	....	....
August	304.8	272.0	....	....	....
September	311.9	288.2	....	....	....
October	301.9	183.5	....	....	....
November	227.9	164.4	....	....	....
December	221.1	192.0	....	....	....

* To date (May 27, 1931).

CHAPTER XI

MULTIPLE AXIS GRAPH

Figure 19

PURPOSE

The purpose of the multiple axis graph is to facilitate comparison of the direction and timing of the changes in two or more series. It differs from the multiple scale graph in that each series fluctuates around its own base, and is expressed in percentages of a base set equal to 100. For example, series for stock and bond prices are related to a par value of 100, or again index numbers, such as those for prices or production, often fluctuate above and below 100 as normal. The 100 line accordingly is used as an axis. If several series are to be compared, plotting on the same grid might result in interlacing of the curves, thus causing confusion or obscurity. For this reason separate axes, each representing 100, are used in order that the picture may be made clearer by separating the plotted lines. In some cases, the data may require the use of the zero lines as axes, but this is exceptional.

CONSTRUCTION

Arithmetic vertical scales should be used if the *amount* of change is to be observed, and logarithmic scales if the *rate* of change is the essential factor to be compared. Use of logarithmic scales excludes the use of the zero line as an axis. The horizontal scale is usually the time scale and is common to all the curves.

The simplest way of constructing the multiple axis graph is to plot each series of data separately, using the same size scale for each. The separate curves may be assembled most easily into one graph by taking off each curve successively on a single piece of tracing cloth. The curves can be arranged in any desired sequence on the tracing because the order and location may be shifted by moving the tracing cloth. This procedure obviates the necessity of planning in advance the most desirable order.

Frequently this can be determined better after the data have been plotted and the curve movement noted.

When the 100% line is the important line, it should be heavy. Usually no horizontal grid lines except the axes are included, as the primary consideration is the relation of the curves to each other. If the zero line on an arithmetic scale is used as an axis, this line should be made heavy.

The essentials of construction are not different from those already described for the arithmetic line graph. It is preferable to turn the 8½ by 11 inch paper so that the longer dimension is vertical.

INTERPRETATION

The interpretation of a multiple axis graph is clear if one bears in mind that its purpose is to facilitate the comparisons of the time and amplitude changes in the curves

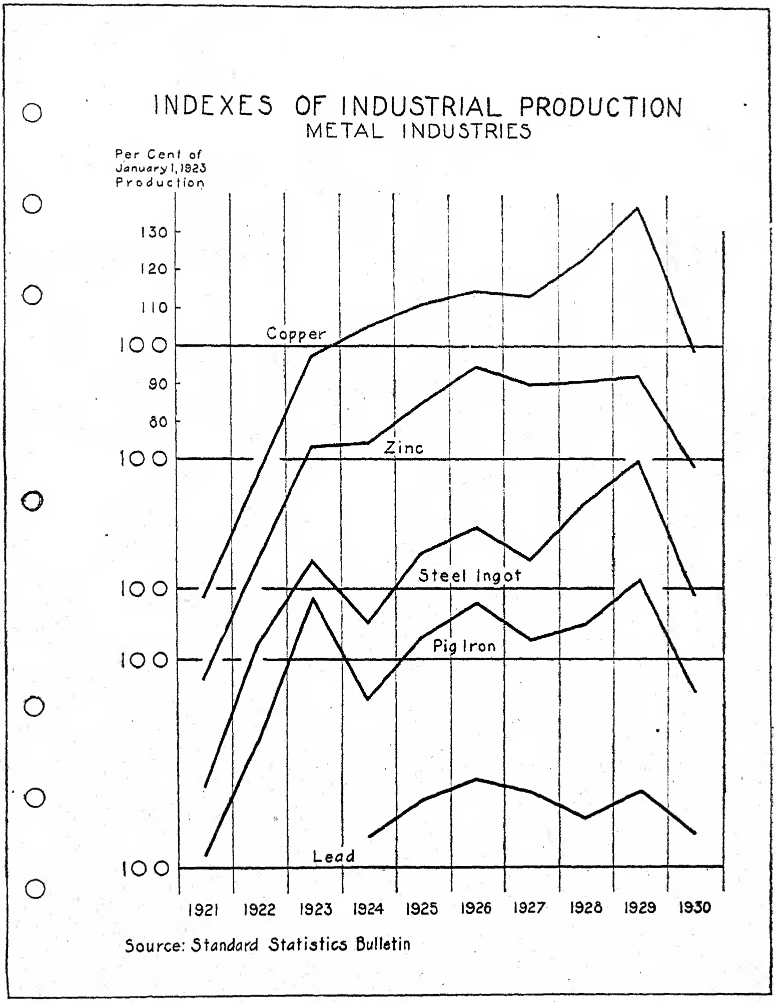

INDEXES OF INDUSTRIAL PRODUCTION
METAL INDUSTRIES
Per Cent of January 1, 1923 Production
130
120
110
100
90
80
Copper
Zinc
Steel Ingot
Pig Iron
Lead
1921
1922
1923
1924
1925
1926
1927
1928
1929
1930
Source: Standard Statistics Bulletin

Fig. 19.

Data to Accompany Multiple Axis Graph

The following data represent the Standard Statistics Indexes of Industrial Production for five metals. They are corrected for normal seasonal variation, but not for secular trend. They are expressed as relatives, with the value for January 1, 1923 equal to 100.

Year	Pig Iron Production	Steel Ingot	Lead	Copper	Zinc
1921	47.9	47.4	.....	33.5	42.0
1922	77.6	85.2	.....	66.0	72.7
1923	155.6	107.1	.....	97.2	103.0
1924	89.6	90.5	107.8	104.7	104.1
1925	105.3	109.0	117.3	110.5	114.8
1926	112.8	115.6	122.7	114.5	124.1
1927	104.6	107.0	119.2	112.9	119.4
1928	108.8	122.1	112.5	122.9	120.2
1929	121.2	133.5	119.4	136.5	122.0
1930	90.7	98.0	108.5	98.3	97.9

CHAPTER XII

CORRELATION OR DOT GRAPH

Figure 20

PURPOSE

This graph derives its name from the fact that it is used to picture the correspondence in changes or the covariation in two selected series of data. As in the case of most types of graphs, each point marked on the graph is determined by two numerical values. In this case, however, one of these values is selected from one series of data and the other corresponding value from the other series of data. The use of the word "corresponding" immediately implies some connecting link between the two series. This point is discussed in Chapter IX, Book II. The real value of the correlation or dot graph lies in the fact that it presents a picture of the relationship between the two series of data. With this fact in mind, it is frequently easy to determine from the graph whether the relationship is linear or curvilinear and also to determine approximately how well a particular type of line or curve fits the data. As a practical business tool for the executive, the dot graph is vitally important in many problems.

CONSTRUCTION

In the construction of this graph an arithmetic grid ordinarily is used, although a semilogarithmic or double logarithmic grid is of value at times. The principles used in constructing the grid and in plotting the points for the correlation graph are the same as those described in Chapter III for the arithmetic line graph. The plotting of each point requires two values, one of which will be selected from one series, while the other will be the corresponding value from the other series. After the data have been plotted, it is necessary to examine the plottings as a group to find whether there is concentration along some path. If such a concentration is apparent, then a line should be drawn through the group of plotted points in such a way that it represents the general tendency

of the concentration. Whether the line is straight or curved will depend on the distribution of the clustered dots. The line selected is referred to as the line of average relationship (see Chapter IX of Book II).

If the position of a straight line of relationship is determined mathematically, it can be located on the graph as described on pages 190–192.

When no path of concentration has been observed, no correlation is present. In such a case no line should be drawn.

INTERPRETATION

Little need be said here in regard to the interpretation of the graph as this is covered fully in the chapter on Correlation in Book II.

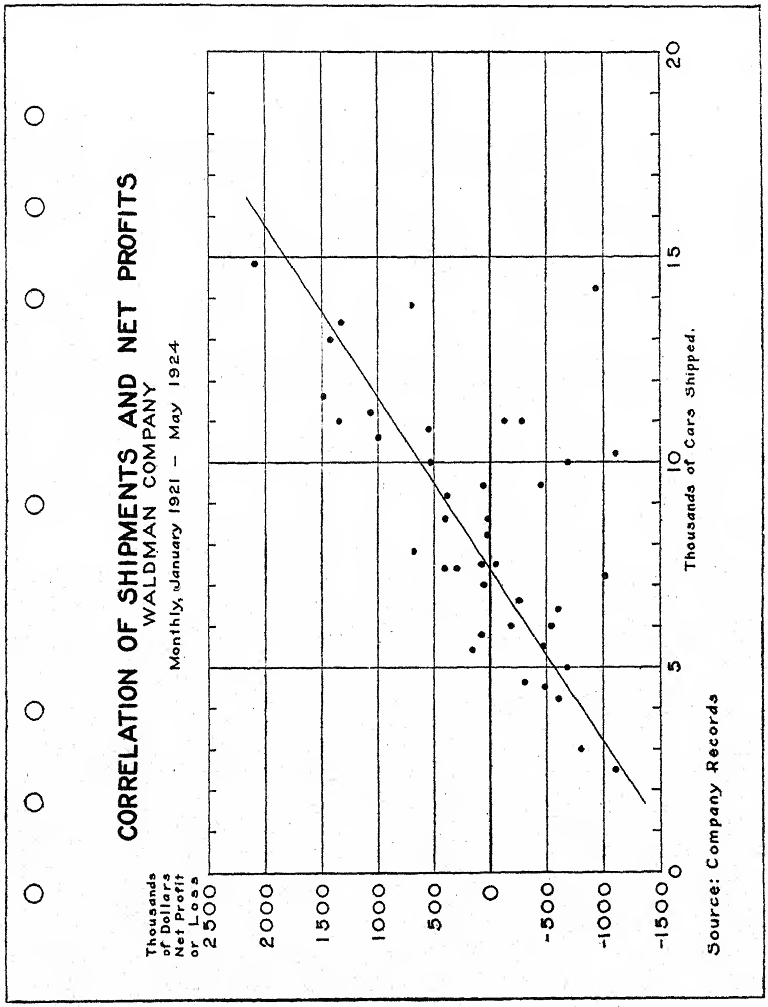

CORRELATION OF SHIPMENTS AND NET PROFITS
WALDMAN COMPANY
Monthly, January 1921 – May 1924
Thousands of Dollars Net Profit or Loss
2500
2000
1500
1000
500
0
-500
-1000
-1500
0
5
10
15
20
Thousands of Cars Shipped.
Source: Company Records

Fig. 20.

Data to Accompany Correlation Graph

The following tabulation is from the records of the Waldman Company,* manufacturers of automobiles, and was used to prepare the attached correlation graph.

Date	Shipments, Hundreds of Cars	Net Profit or Loss, Thousands of Dollars	Date	Shipments, Hundreds of Cars	Net Profit or Loss, Thousands of Dollars
1921			1923		
January....	110	1,340	January ...	74	290
February...	70	45	February ..	78	680
March	86	403	March.....	112	1,070
April.......	108	540	April.......	148	2,090
May.......	94	56	May.......	116	1,490
June.......	60	−180	June.......	100	520
July........	55	−470	July.......	92	370
August.....	82	20	August	86	20
September .	75	−50	September.	110	−120
October....	30	−800	October ...	94	−450
November..	45	−490	November .	72	−1,010
December..	25	−1,100	December .	74	400
1922			1924		
January....	42	−600	January ...	100	−680
February...	50	−690	February ..	102	−1,110
March	75	70	March.....	142	−940
April.......	106	1,000	April.......	110	−280
May.......	130	1,420	May.......	58	70
June.......	134	1,320	June	57	....
July........	46	−300	....	...	....
August.....	138	700	....	...	....
September .	64	−600	....	...	....
October....	60	−540	....	...	....
November..	66	−260	....	...	....
December..	54	160	....	...	....

* See Brown, T. H., *Problems in Business Statistics,* p. 407.

CHAPTER XIII

BELT GRAPH

Figure 21

PURPOSE

A belt graph represents changes in the components of a series over a period of time by use of colored or shaded zones or belts. This type of graph is an outgrowth of the component column graph, in which the sections of the column are used to display the size of the components. When changes from one time to the next are to be emphasized, the columns are placed contiguous to each other, so that each column is assumed to represent a time unit. When the midpoint values of the components in one column are connected by straight lines with corresponding components in adjacent columns, and when the spaces between the resulting broken lines are shaded, we have a belt graph. The purpose of the belt graph is to present a picture of the way in which the components are changing from time to time.

We have a special case of the belt graph analogous to the special case of the component column graph when we assume the total height of the grid represents 100%. For many uses this special form of belt graph is extremely desirable, since it takes out the variable element which is the changing size of the total.

CONSTRUCTION

The arithmetic grid only can be used. At each midpoint between the boundary lines for the time unit used, data representing each total are plotted. The component data are plotted below their respective total points.

Ordinarily, the largest of the component series should be placed at the bottom of the time interval column, with the others above in decreasing order of size, exactly as in the case of the component column graph. This will avoid a top-heavy appearance. A "miscellaneous" component ordinarily should be placed on top regardless of size. It is obvious that there must be a value

for every component in each time interval and that a common sequence of arrangement must be followed in plotting so that, when points are connected, the belts will represent continuous changes in the component.

Both horizontal and vertical grid lines should be inked lightly across the entire field. Plotted lines should be inked in boldly to serve as boundaries between belts.

Differentiation between belts is secured by colors or by black and white shading. Identification should be made by means of key or legend. No labels should be placed within the belts, since they interfere with the area concept.

INTERPRETATION

The interpretation of the graph inevitably is tied up with its purpose. As has been indicated above, it is preferable to use this graph when changes in the components of a series over a period of time are the important consideration. Since the areas depend in part upon the size of the interval chosen as the time unit, the eye should not be deceived by the size of the shaded areas. The slopes of the bands at the top of the graph reflect changes in the components below as the values are cumulated for each time interval. Thus, in cases where the components are changing rapidly the graph is likely to give a distorted picture. The effect of changing the size of the total may be removed by using the 100% graph referred to above, but the danger of distortion from radical changes in values of components remains.

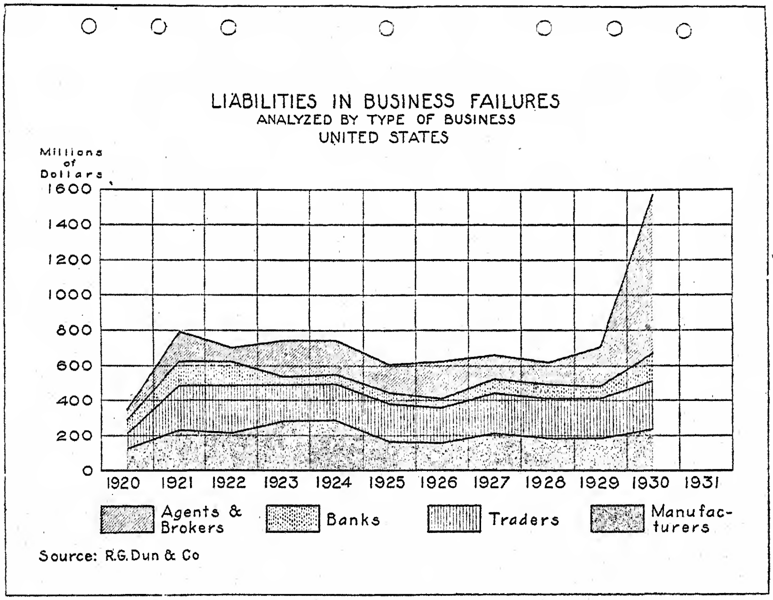

LIABILITIES IN BUSINESS FAILURES
ANALYZED BY TYPE OF BUSINESS
UNITED STATES
Millions of Dollars
1600
1400
1200
1000
800
600
400
200
0
1920 1921 1922 1923 1924 1925 1926 1927 1928 1929 1930 1931
Agents & Brokers
Banks
Traders
Manufacturers
Source: R.G. Dun & Co

FIG. 21.

Data to Accompany Belt Graph

The following information from *Dun's Review*, published by R. G. Dun & Company, shows the liabilities in business failures for 11 consecutive years. Since the figures are given by four classes of business as well as by totals, an analysis can be made in graphic form.

(Thousands of Dollars)

Year	Manu-facturers	Traders	Agents and Brokers	Banks	Total
1920	127,992	88,558	78,571	50,708	345,829
1921	232,907	254,794	139,700	167,849	795,250
1922	214,925	271,388	137,583	77,735	701,631
1923	281,316	209,930	48,140	203,739	743,125
1924	286,770	203,190	53,265	202,926	746,151
1925	167,684	215,368	60,690	164,698	608,440
1926	158,042	201,333	49,856	212,074	621,305
1927	211,504	228,194	80,405	143,449	663,552
1928	182,478	225,301	81,780	129,649	619,208
1929	186,734	224,731	71,784	218,796	702,045
1930	238,639	272,930	156,714	908,157	1,576,440

CHAPTER XIV

DIFFERENCE GRAPH

Figure 22

PURPOSE

The name indicates that the purpose of this graph is to present the difference between two series of data. Since the graph is to be used only when such differences are significant, the two series of data must be in the same units and plotted on the same scale.

CONSTRUCTION

Both the vertical and horizontal scales are arithmetic. The arrangement of scales, field, and other details follows ordinary arithmetic scale line graph practice. Grid lines run through the areas between plotted lines.

Each series is plotted in the usual manner and the points connected in sequence. The area between the two lines represents the difference and is emphasized by the use of colors or shading. Where the curves cross, different colors or shading should be used to throw into relief this reversal in relationship, and a key or legend provided to indicate the significance of this difference.

The two plotted lines should be labeled directly and differentiated in the usual manner in order to follow their fluctuations more easily.

INTERPRETATION

In the interpretation of the difference graph, the significant fact is the difference in vertical distance between corresponding points. The difference also may be measured in areas. There is the danger, however, of getting an erroneous impression. Wherever the two curves form a steeply inclined strip of area it may seem at the first glance that the difference is smaller than it actually is. This is caused by a natural tendency of the observer to measure the area by the shortest route across the area instead

of measuring the total area for the period in question. To avoid this difficulty it is better to confine oneself to the observation of vertical distances only, rather than of areas. The areas are colored or otherwise marked to indicate which of the two lines for a given period of time happens to be uppermost. The significance of the graph depends upon its usefulness in interpreting the relative position of the two series over a period of time.

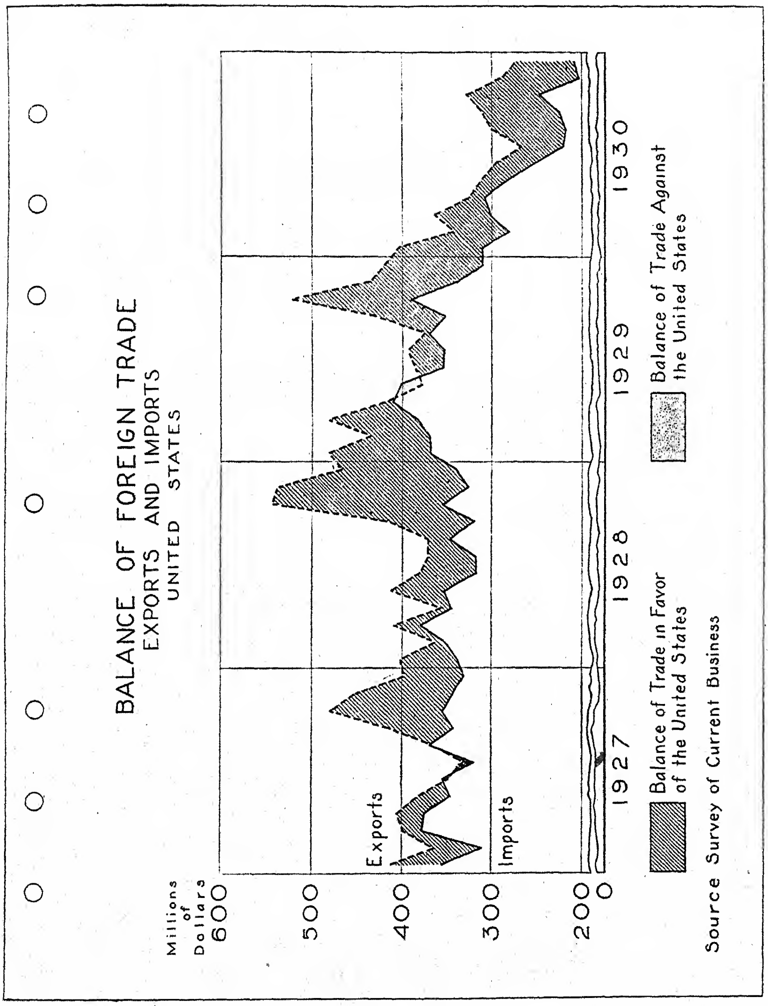

BALANCE OF FOREIGN TRADE
EXPORTS AND IMPORTS
UNITED STATES
Millions of Dollars
600
500
400
300
200
0
Exports
Imports
1927
1928
1929
1930
Balance of Trade in Favor of the United States
Balance of Trade Against the United States
Source Survey of Current Business

Fig. 22.

Data to Accompany Difference Graph

The series below show total value of exports and total value of imports for the United States, by months, during 1927, 1928, 1929, and 1930. They are compiled by the U. S. Department of Commerce and published in the *Survey of Current Business*.

(Thousands of Dollars)

Date	Imports	Exports	Date	Imports	Exports
1927			1929		
January	356,841	411,649	January	368,897	480,384
February	310,877	364,613	February	369,442	434,529
March	378,331	398,246	March	383,818	481,710
April	375,734	405,001	April	410,666	418,051
May	346,501	382,385	May	400,149	377,083
June	354,892	348,546	June	353,403	386,799
July	319,298	332,994	July	352,981	393,798
August	368,820	367,575	August	369,358	374,723
September	342,154	416,472	September	351,304	431,801
October	355,744	480,347	October	390,998	522,380
November	344,267	452,023	November	338,473	435,527
December	330,920	398,377	December	310,573	420,622
1928			1930		
January	337,916	401,913	January	310,968	404,377
February	351,035	362,614	February	281,707	342,964
March	380,437	409,961	March	300,460	363,162
April	345,314	356,057	April	307,824	326,545
May	353,981	413,829	May	284,683	312,592
June	317,249	380,305	June	250,343	289,827
July	317,848	371,471	July	220,494	261,960
August	346,715	371,312	August	218,417	293,899
September	319,618	414,859	September	226,352	307,945
October	355,358	543,171	October	247,322	322,941
November	326,565	538,375	November	203,713	285,441
December	339,408	466,232	December	208,721	270,810

CHAPTER XV

ZEE GRAPH

Figure 23

PURPOSE

The zee graph derives its name from the fact that the plotted lines form roughly a Z-shaped figure. The object is to present in a single graph three different forms of the original data. It is the way in which these data are presented which determines the appearance of the graph. These forms are (1) the original data plotted by periods, for instance, by months, (2) a progressive cumulative total plotted exactly as described in the case of the cumulative graph, (3) a moving *total* of the values: generally a moving total of the immediately preceding 12 months is used. The accompanying illustration shows these details.

CONSTRUCTION

Two arithmetic scales are used because of the divergence in magnitude of the figures. One scale is for the original data and the other scale is for the cumulative data and the moving totals. Ordinarily, if the scale of the monthly data is made five times greater than the scale used for the moving total and the accumulated total, a proper adjustment will be secured.[1]

The lines are plotted on an arithmetic grid. The points are plotted at the end of each space which represents an interval of time, as in the cumulative graph.

Identification of lines and scales follows principles already enumerated.

Customary practice provides a separate graph for each year covered, in order that any desired sequence of years may be readily compared.

[1] For weekly data, where the moving total is 52 weeks, the ratio should be 20 to 1; for daily data, where the moving total is monthly, the ratio should be 10 to 1.

INTERPRETATION

The zee graph enables the reader to interpret the development of conditions within a given business. In the case of the data plotted in Fig. 23 the current monthly sales, total sales for the previous 12 months, and the cumulative sales for the year to date are shown. Not only does the graph show how the current monthly sales have been going and what the cumulative total for the year to date is, but also the graph indicates a comparison between the current monthly figure, the cumulative total for the year to date and the total sales for the previous 12 months. These are all figures which are commonly used in the control of various businesses.

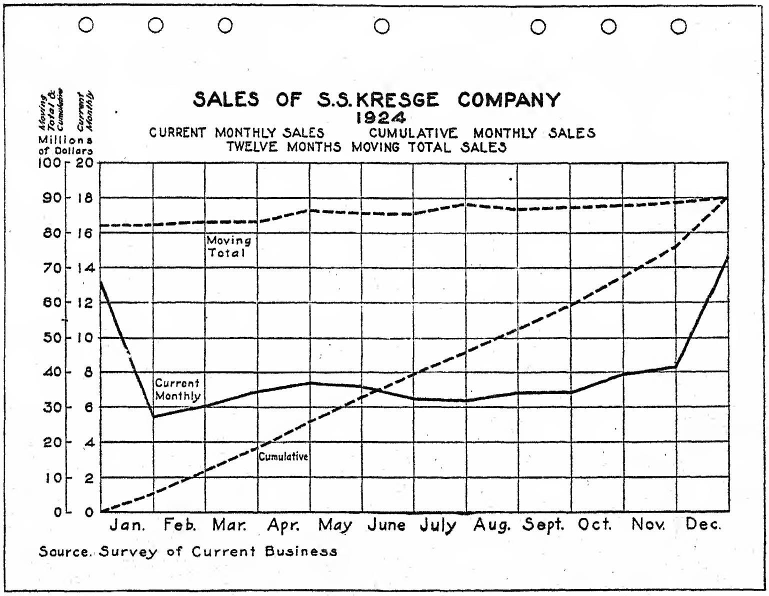

SALES OF S.S. KRESGE COMPANY
1924
CURRENT MONTHLY SALES CUMULATIVE MONTHLY SALES
TWELVE MONTHS MOVING TOTAL SALES
Moving Total & Cumulative
Current Monthly
Millions of Dollars
100 20
90 18
80 16
70 14
60 12
50 10
40 8
30 6
20 4
10 2
0 0
Moving Total
Current Monthly
Cumulative
Jan. Feb. Mar. Apr. May June July Aug. Sept. Oct. Nov. Dec.
Source. Survey of Current Business

FIG. 23.

Data to Accompany Zee Graph

The following tabulation based on data published in the *Survey of Curren Business* shows the necessary information required to build the accompanying zee graph of sales by the S. S. Kresge Company Chain Stores during 1924.

(Thousands of Dollars)

Month	1923		1924		
	Monthly Sales	12 Months' Moving Total Sales	Monthly Sales	12 Months' Moving Total Sales	Monthly Cumulative Sales
January	4,929	......	5,457	82,372	5,457
February	5,016	......	6,019	83,375	11,476
March	6,950	......	6,875	83,300	18,351
April	5,862	......	7,370	84,808	25,721
May	6,370	......	7,157	85,595	32,878
June	6,485	......	6,478	85,588	39,356
July	5,746	......	6,371	88,213	45,727
August	6,338	......	6,802	86,677	52,529
September	6,324	......	6,851	87,204	59,380
October	7,246	......	7,872	87,830	67,252
November	7,508	......	8,252	88,574	75,504
December	13,070	81,844	14,592	90,096	90,096

CHAPTER XVI

MAP GRAPH

Figure 24

PURPOSE

The purpose of the graph is to picture facts in relation to geographic or political areas. There are two types which are important. The first is used to indicate the location of numerical facts without regard to the relative size of the geographical division. The second is used to convey the idea of the relation of numerical facts to the area of various geographical divisions.

An illustration of the first type is the location of sales branches or number of salesmen in a given territory. Each is indicated by a single dot. Figure 24 shows an example of the second type. Here the quantity of the improved land is directly related to area. The shading shows the relative proportion of improved land.

In connection with this second point, sometimes the geographic map of the United States or of the world is distorted by changing the map area of the various states so that their map area is proportional to their relative importance in connection with some selected series of facts. The construction of this type of map is very difficult and the interpretation often misleading because of the deceptiveness of areas.

It is believed that, when the concept of location is not of primary importance, statistical facts in connection with many business problems are better presented through the aid of other types of graphs enumerated in previous chapters.

CONSTRUCTION

Obviously, the basic thing as far as location is concerned is to secure a convenient system of marking the map. Thus, on the map a single dot may represent some selected unit. The number of dots within a given area will represent the clustering in that geographic or political area. Similar results often are

obtained by means of various shadings. This also is a helpful way to convey the idea of quantity with respect to location. Because of the difficulty of drawing accurately the outline of the several states, it is usually more satisfactory to secure outline maps. The map should, of course, carry a title and a legend.

INTERPRETATION

If the simple concept of location is retained, the interpretation is obvious and simple. On the other hand, many map graphs attempt to indicate in addition the idea of magnitude. Consequently, when interpreting one of these graphs, care should be taken that the facts which the map is supposed to present are clearly understood. When the size of the geographical divisions are distorted, the interpretation should be based on the area and not on linear dimensions.

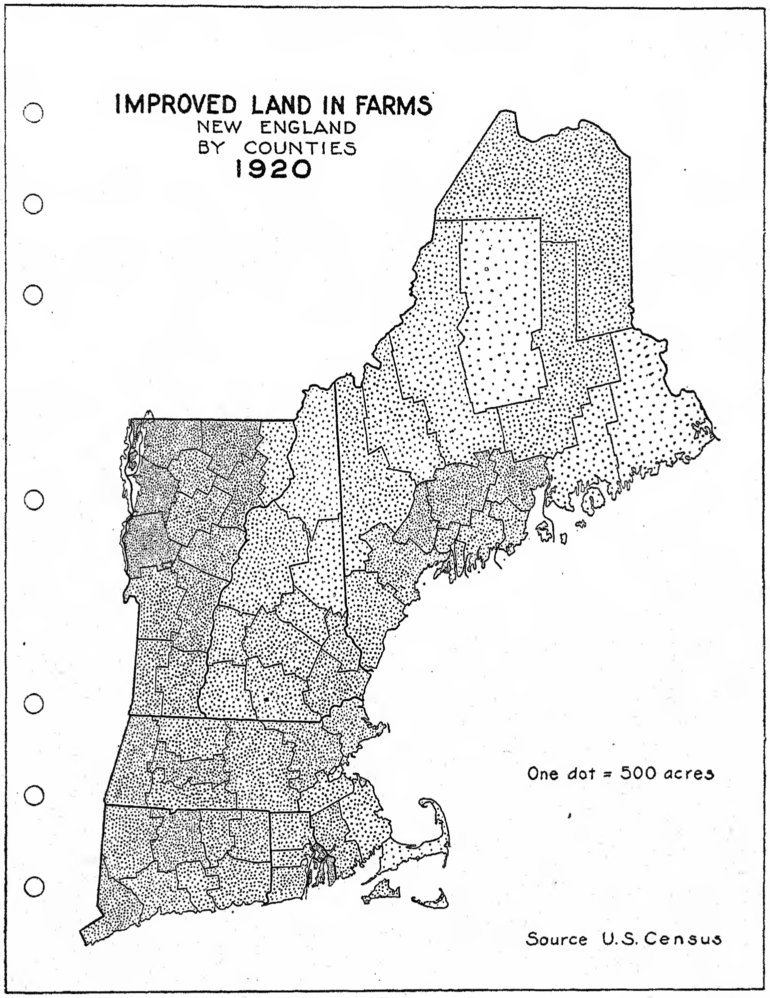

IMPROVED LAND IN FARMS
NEW ENGLAND
BY COUNTIES
1920
One dot = 500 acres
Source U.S. Census

FIG. 24.

Data to Accompany Map Graph

The following tables, taken from the U. S. Census Report, Vol. 6, Part 1, show the acreage of improved land in farms in New England and were used in preparing the accompanying map.

Acreage of Improved Land in Farms, 1920

Location	Acres	Location	Acres
Maine:	1,977,329	New Hampshire:	702,902
Androscoggin County	90,483	Belknap County	39,377
Aroostook County	450,763	Carroll County	45,431
Cumberland County	129,454	Cheshire County	53,678
Franklin County	96,294	Coos County	90,530
Hancock County	44,896	Grafton County	131,827
Kennebec County	173,835	Hillsborough County	77,286
Knox County	34,048	Merrimac County	84,078
Lincoln County	51,524	Rockingham County	86,336
Oxford County	134,722	Strafford County	41,436
Penobscot County	219,485	Sullivan County	52,923
Piscataguis County	67,880		
Sagodahoc County	33,868	Massachusetts:	908,834
Somerset County	180,315	Barnstable County	13,619
Waldo County	117,424	Berkshire County	139,744
Washington County	56,347	Bristol County	68,061
York County	95,991	Dukes County	7,790
		Essex County	64,429
Vermont:	1,691,595	Franklin County	75,307
Addison County	217,796	Hampden County	73,825
Bennington County	81,691	Hampshire County	90,083
Caledonia County	129,997	Middlesex County	117,290
Chittenden County	140,453	Nantucket County	1,572
Essex County	38,817	Norfolk County	30,183
Franklin County	150,287	Plymouth County	44,101
Grand Isle County	33,141	Suffolk County	1,663
Lamoille County	73,547	Worcester County	181,167
Orange County	123,999		
Orleans County	173,646	Connecticut:	701,086
Rutland County	165,368	Fairfield County	108,393
Washington County	106,639	Hartford County	142,506
Windham County	96,904	Litchfield County	135,616
Windsor County	159,310	Middlesex County	35,357
		New Haven County	75,880
Rhode Island:	132,855	New London County	79,839
Bristol County	5,413	Tolland County	53,024
Kent County	14,712	Windham County	70,471
Newport County	29,794	....	......
Providence County	41,646	....	......
Washington County	41,290	....	......

CHAPTER XVII

PIE GRAPH

Figure 25

PURPOSE

This graph is a disc divided into sectors so that it resembles a product of the culinary art. Pie graphs are commonly used for two purposes. One is a comparison of sectors which make up a total represented by the disc. The other is a comparison of the sizes of two or more totals, with the sectors in each pie representing the subdivisions of each total.

CONSTRUCTION

Theoretically, all that is necessary in connection with the use of a pie graph is a circle divided into 100 parts, but practically there has to be a transposition of percentages into degrees as protractors divide the circle into 360° instead of 100 parts. Printed circles divided into 100 parts are available, but protractors with that division are very difficult to find. This means that each part will correspond to a unit of 1% of whatever total the statistician desires to present. There is no general rule as to the arrangement or sequence of the sectors. The sectors may be labeled according to the original quantities or each sector may be labeled as a percentage of the whole. The latter form is more common.

The various sectors should be differentiated by the use of colors or by the use of cross-hatching with ink. A key or legend should be added to show the meaning of various kinds of cross-hatching if direct labels are not used.

INTERPRETATION

For many purposes a pie graph is satisfactory, since the interpretation is simple and direct. Difficulty in interpretation arises when two or more discs of different sizes are compared. Frequently the comparison is made on the basis of a direct relation between the *diameters* of the two circles; thus, if one pie is to be twice the size of the other, the diameter of one is made twice the size of the other. Correct construction would make the *area*, not the diameter, of one twice the size of the other.

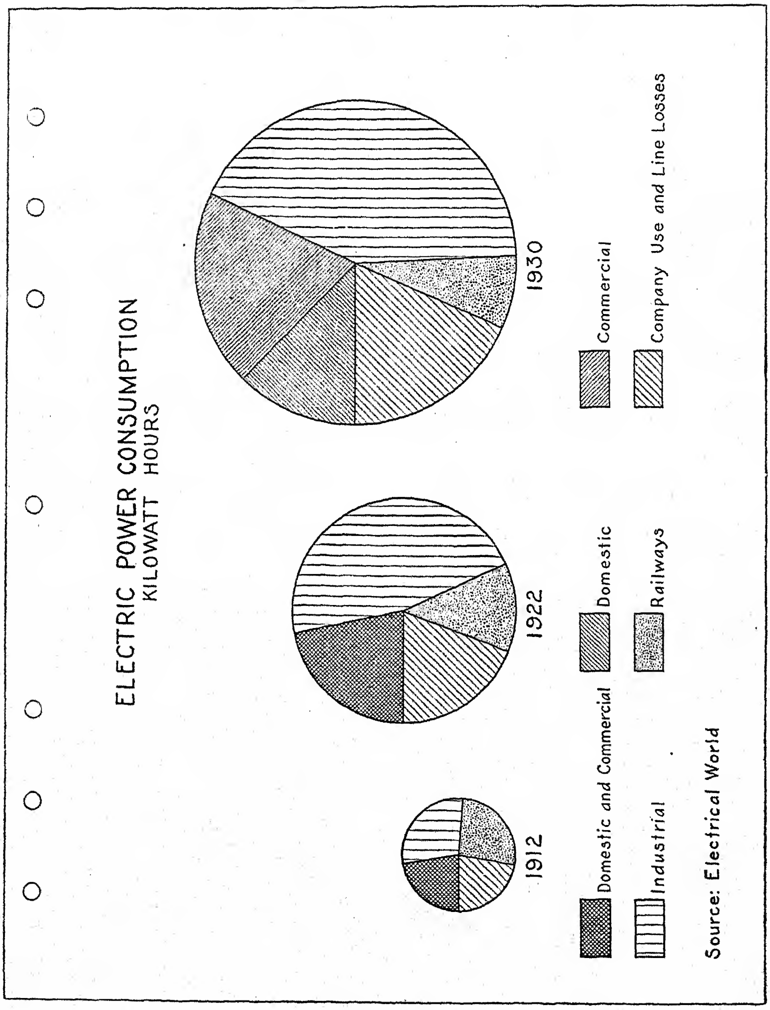

ELECTRIC POWER CONSUMPTION
KILOWATT HOURS
1912
1922
1930
Domestic and Commercial
Domestic
Commercial
Industrial
Railways
Company Use and Line Losses
Source: Electrical World

Fig. 25.

Data to Accompany Pie Graph

Printed below are data from the *Electrical World*, January 3, 1931. These figures represent the consumption of power by certain classes of customers. It is to be noted that the figures for 1930 give one more classification than those for the previous years. The accompanying pie graph is constructed upon an area basis.

(Millions of Kilowatt-hours)

Sales	1912	1922	1930
Domestic } Commercial }	2,572	9,560	{ 11,640 { 18,370
Industrial	3,254	20,620	39,300
Railways	3,017	5,642	6,810
Company Use and Line Losses	2,546	8,600	16,906
Total	11,389	44,422	93,026

CHAPTER XVIII

EXERCISES AND PROBLEMS

EXERCISES[1]

Chapter III. Arithmetic Scale Line Graph

Plot the annual figures for the last 20 years of the following series: Wholesale commodity price index of the United States Bureau of Labor Statistics.
Source: Bureau of Labor Statistics, Department of Labor.

Chapter IV. Logarithmic Scale Line Graph

Plot the monthly figures for the last four years of the two following series: Bank debits in New York City, and bank debits outside New York City. Comment on the relative changes in the two curves.
Source: Federal Reserve Bulletin.

Chapter V. Bar Graph

Make a bar graph for the annual figures for the last five years of the two following series: Anthracite production and bituminous coal production. Comment on the comparison.
Source: Bureau of Mines, Department of Commerce.

Chapter VI. Component Column Graph

Make a component column graph for the last five years of United States exports to Europe, Asia, North America, South America, Oceania, and Africa. Comment on what the graph indicates.
Source: Foreign and Domestic Commerce, Department of Commerce

Chapter VII. Frequency Graph

Organize the total expense figures for 180 department stores, found on pages 101–103 of this Handbook into a frequency distribution, and plot the distribution.

Chapter VIII. Cumulative Graph

Plot on a grid arranged for twelve months the cumulative data for each of the last two full years and for as much of the current year as is available

[1] The data for these exercises may be found in *Standard Statistical Bulletin* in addition to the primary source indicated.

of the following series: Montgomery Ward & Company sales. Comment on the present condition of sales as indicated by the chart.

Chapter IX. Multiple Scale Graph

Plot on a multiple scale grid monthly data for the last three years for production, stocks, and wholesale prices of leather. Write a paragraph commenting on the relationship of the three curves.

Source: Production and Stocks: Department of Commerce. Prices: *Hide and Leather.*

Chapter X. Stock Market Graph

Select a stock the price of which is quoted in daily papers or some financial periodicals. Plot the weekly prices and numbers of shares traded for the period of the last three months. Comment on the changes in both series.

Source: Annalist

Chapter XI. Multiple Axis Graph

Choose five related series of index numbers. Plot their monthly values for the last three years on a multiple axis grid. Comment on any facts brought out by the comparison.

Chapter XII. Correlation or Dot Graph

Using monthly figures for the last three years, make a dot chart showing the relationship between raw sugar receipts and raw sugar prices. What is the direction of the line of average relationship? Is the indicated correlation close?

Source: Statistical Sugar Trade Journal.

Chapter XIII. Belt Graph

Plot the annual average figures, for the last five years, of monthly finished cotton goods stocks, including total stocks as made up of white goods, dyed goods, and printed goods stocks. Is this type of graph helpful in making a comparison of the four time series?

Source: Association of Cotton Textile Merchants.

Chapter XIV. Difference Graph

Make a difference graph showing annual totals for the last ten years of gold exports and imports. Comment on the balance of the gold movement.

Source: Federal Reserve Bulletin.

Chapter XV. Zee Graph

Make a zee graph using the monthly figures for the last complete year of total corporation dividend payments. Write briefly on what the graph shows.

Source: Standard Statistical Bulletin.

Chapter XVI. Map Graph

Using an outline map of the United States, plot total motor vehicle registration by states.

Source: National Automobile Chamber of Commerce.

Chapter XVII. Pie Graph

Make a pie graph using annual figures for the last year of United States imports from Europe, Asia, North America, South America, Oceania, and Africa.

Source: Bureau of Foreign and Domestic Commerce, Department of Commerce.

PROBLEMS

In the following exercises the students are expected to decide on the appropriate type of graph.

Automobile Production, Passenger Cars. Plot the monthly figures for the last two years in such manner as to show at the end of each month the production "to date" in one year compared with production for the same period in the previous year.

New Corporate Security Issues, the total, stocks, and bonds and notes, showing the annual totals for the last three years. Discuss the changes in the proportions of the component parts.

Road Building, Cement Production, and Cement Prices. Plot monthly figures for the last two years. Discuss the relationship.

Cotton Production and Cotton Prices. Plot annual figures for the last 15 years. Discuss the relationship.

Total Sales of Two Mail Order Houses and Total Sales of Four Ten-cent stores. Plot monthly averages for the last 10 years. Compare and discuss the relative growth in the two series.

Productive Activity in Manufacturing Plants in United States. Plot the figures for the last full year for each of the following series: All Industry, Automobiles, Food and Kindred Products, Rubber Products, and Shipbuilding. Compare the first series with the other four, and discuss the results of the comparison.

Bank Debits Index, and the Index of Sales of the Five- and Ten-cent Stores. Plot the monthly figures corrected for seasonal variation for the period of last three years. Write a paragraph commenting upon the comparison.

General Motors Corporation Sales to Dealers, and Dealers' Sales to Users. Plot monthly figures for the last three years. Discuss whether the graph indicates any changes in the coordination of production and consumption.

Plot the yearly ranges of 40 domestic bond prices for the last 12 years (published by the *New York Times*). Indicate the years in which the bond prices declined. What have been the changes in relative fluctuations in prices?

BOOK II

MATHEMATICAL METHODS

CHAPTER I

INTRODUCTION

The growing use of statistics in connection with the analysis of business problems or the control of business enterprises frequently makes necessary extended numerical calculations. Too often such calculations seem to consist of an endless variety of methods. The methods themselves, however, can be grouped under one or more of four major divisions. These are: (1) frequency distributions and the theory of probability, (2) index number construction, (3) time series analysis, and (4) correlation. In the following chapters necessary calculations are presented for each of the elementary methods described.

In the development of the work the steps in each calculation are enumerated so that the student with some thought should be able to work out similar calculations. In order that a proper understanding of the interrelation between the calculations and the various phases of the business problem may be acquired, each group of illustrative calculations is developed as the solution of a business problem. The calculations should be studied, therefore, in the light of the conditions set up by each of these problems.

Although the necessity of ability to make statistical calculations is recognized, such calculations made mechanically possess no value in themselves. The use of methods always is the result of the demand of some particular problem for a certain type of statistical summary which will give the information desired for that problem. A definite knowledge of the objective in each problem must be added to a clear understanding of the technical detail and the philosophical background.

CHAPTER II

FREQUENCY DISTRIBUTIONS

The words "frequency distribution" may picture to the individual the prospect of a rather prosaic, technical routine to be applied to certain statistical problems. Actually, however, in many events of everyday life, statements of facts implying frequency distributions are commonly used. We have the habit of saying that "usually" or "on the average" certain things are true. Thus: "On the average it takes 20 minutes to walk down town." "Usually the air is clear in the mountains." "Ordinarily I eat a light breakfast." "Our office hours are from 8:30 to 5, but usually we do not get away until after that." Sometimes the statements become technically more definite. Thus, in baseball such terms as batting average or fielding average are well known and understood.

By the statement that it usually takes us 20 minutes to walk from our home to the office, we do not mean that it always takes us exactly 20 minutes, no more and no less. We know very well that at times it takes us a little more and at other times a little less than 20 minutes. Likewise, we know that a worker on piece rates in a factory will not earn always exactly the same amount, and that not all workers performing the same operations will earn the same amount. Some earn a little more and some a little less than the amount we call the average. In other words, the values which represent the individual records in any one given performance are distributed above and below the typical value which we have in mind when we say "usually" or "average." At times another idea must be included to modify our words "usually" or "average." We hear such statements as, "John gets home for dinner at 6:30. He is almost never more than five minutes late." "Henry is so busy at the office that he often has to stay after hours. He should be home by quarter past six, but frequently he does not arrive until quarter of seven or seven o'clock." Often in athletic contests we speak of the closeness of the results of trials by the term "consistency of performance," and say that a man is consistently good or bad within a narrow

range or, on the other hand, another man's performance is likely to be very erratic. This additional measure of a frequency distribution is known technically as dispersion. It represents the scatter of values about the average value. Average and dispersion illustrate the major facts that we need to know about frequency distributions. We will now discuss the steps that lead to the knowledge of how to calculate these two and certain other measures.

If we should keep a day-to-day record of the time that it takes us to go from our home to the office, we would have the record of figures arranged chronologically but not necessarily in order of size. A stormy day, the meeting of a friend, the missing of a trolley or subway train, might well make one particular day's trip longer as compared with preceding or following trips. Furthermore, the variations of time would be likely to occur irregularly.

Since little can be learned from a disorganized mass of information, the first problem in analyzing data is the arrangement of the figures according to size.

This problem of the orderly arrangement of data can be compared to that of grading different sizes of coal, where one grade is put in one bin, the next in another, and so on. Of course, if one went to the trouble of weighing each lump with a chemical balance, it would be entirely possible to arrange lumps of coal precisely in order of size so that each lump would be heavier than the one just before it and lighter than any which followed. Such a method with a large amount of coal would be perfectly hopeless. Consequently, for obvious practical purposes, rather broad standard groups or sizes are used when marketing coal or other similar commodities. Figures are grouped in exactly the same way when frequency distributions are constructed.

If the figures are arranged in order of size by groups, we commonly find that the sorting distributes the figures in a way similar to that shown in the tabulation on the chart, Fig. 27. When this is done, certain information in regard to the numbers can be derived. Usually the identity of the individual figures in the group may be omitted so that only the actual number of items appearing in each group need be used.

The regularity of the distribution is affected to a certain degree by the number of classes or groups selected. The number should be determined by trial and error. A good working rule is that

there should be from 15 to 20 groups for any one frequency distribution.

After the data for a problem have been grouped as described, it will be discovered that in general the arrangement will be similar to that already described above for Fig. 27; that is, the plotted columns will appear to be arranged in the form of a more or less symmetrical, bell-shaped diagram.

When the two sides of the bell-shaped diagram are symmetrically arranged and when the proportions between heights of the various columns with reference to their position have a certain relationship as pictured in Fig. 26, we have a distribution which is called a normal distribution. We shall see below, page 93, however, that not all symmetrical distributions are necessarily normal. The nonsymmetrical or skew distributions are one-sided distributions, similar to those shown in Figs. 27 and 28. In these unbalanced distributions the larger side may come on the right or on the left of the largest group which includes the peak.

The above general statements apply to the appearance of frequency distributions when the first step of grouping and plotting the data has been completed. For the purpose of describing the characteristics of the distribution curve, there are five measures commonly in use. These are (1) the average, (2) the dispersion, (3) the kurtosis, (4) the skewness, and (5) the displacement.

1. *The average* is used to indicate the typical figure and to describe the concentration or central tendency of items. Of the various types of averages, the arithmetic mean or arithmetic average is the one most commonly used. Since it is the one frequently referred to by the word "average," some people are accustomed to think of it as the only average. The arithmetic average, however, is one of several types. The *mode* is another type of average. It is that value on the horizontal scale which corresponds to the largest group or maximum ordinate in a frequency distribution. The *median* is still another kind of average which indicates the point on the scale of values that divides the whole distribution into two equal parts. When the distribution is symmetrical, these averages have identically the same value, but when it is not symmetrical, the three figures are different. The spread of these measures is indicated in Fig. 27. Although other measures of central tendency of frequency distributions are known, they are not as commonly used, and hence will not be described here.

2. *The dispersion* is measured by the amount of variation of the items from their central tendency, usually the arithmetic average. The difference between the two extreme figures is known as the *range*. Since the dispersion usually is measured by the average scatter, it is affected by the concentration of the items, as well as by the difference between the extremes. A clustering of the items about the average indicates a consistency in performance, or a narrow dispersion. We use this idea in everyday life when we say that an individual is consistent in his performance in any particular activity. The opposite condition of inconsistent or erratic performance denotes a wide dispersion.

The four curves given in Fig. 26 illustrate different distributions of weekly wages in four different companies. In all four cases the modal wage is \$30, which was the wage of the largest group of men in each shop.

CASE 1

This case and Case 2 are examples of series which form normal distributions. In Case 1 most of the men receive \$30 a week, but some of the men receive wages which differ from the \$30 by as much as \$12. Thus the range of variation is \$24. Since the distribution of wages is symmetrical, the mode, median, and arithmetic mean all have the same value of \$30.

CASE 2

The largest group of men here also receives \$30 a week, but those who receive a wage different from \$30 a week are relatively few. None of them differs from the \$30 wage by more than \$4 above or below that amount, so that the range is only \$8. The dispersion is therefore small. Since the distribution is symmetrical, the arithmetic average, the median, and the mode are all \$30.

3. *Kurtosis* describes the degree of "flat-toppedness" or the peakedness of a distribution.

CASE 3

In Case 1 and Case 2, given above, the distributions were symmetrical and normal. Curves, however, may be symmetrical, but not normal. Case 3 illustrates this. The flat-toppedness

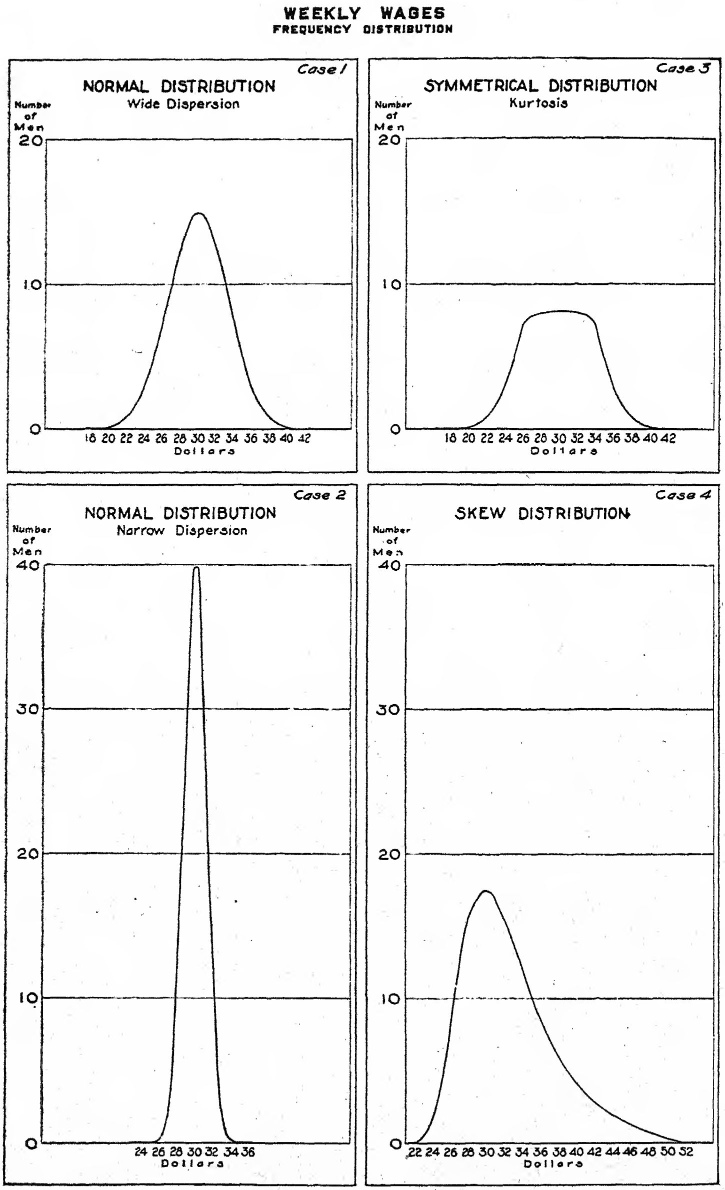

WEEKLY WAGES
FREQUENCY DISTRIBUTION
Case 1
NORMAL DISTRIBUTION
Wide Dispersion
Number of Men
20
10
0
18 20 22 24 26 28 30 32 34 36 38 40 42
Dollars
Case 3
SYMMETRICAL DISTRIBUTION
Kurtosis
Number of Men
20
10
0
18 20 22 24 26 28 30 32 34 36 38 40 42
Dollars
Case 2
NORMAL DISTRIBUTION
Narrow Dispersion
Number of Men
40
30
20
10
0
24 26 28 30 32 34 36
Dollars
Case 4
SKEW DISTRIBUTION
Number of Men
40
30
20
10
0
22 24 26 28 30 32 34 36 38 40 42 44 46 48 50 52
Dollars

FIG. 26.

of the curve as pictured illustrates kurtosis. It indicates the deviation of a symmetrical curve from normal.

4. *Skewness* refers to the lack of symmetry in the distribution. It indicates that more items are to be found on one side of the largest group of items than on the other side. In everyday life we say that, although a person usually walks from his home to the office in 20 minutes, he is more likely to take a longer rather than a shorter time. Of course, in this case the distribution would taper off toward the longer period. The chart, Fig. 27, referred to above, shows a skew distribution.

CASE 4

In this case, also, the largest group of men receives $30 per week, but those who receive a wage different from $30 are relatively numerous. The graph shows that the men who earn over $30 outnumber those who earn less. This distribution is skewed to the right. When this condition exists the median is located away from the mode, toward the right. The arithmetic mean being influenced even more than the median by the extreme items is found beyond the median toward the right. In case of a distribution skewed in the opposite direction, mode, median, and arithmetic mean would be arranged in the opposite order to the left.

5. *Displacement* is a measure of internal shifting of the individual items when such shifting does not change the shape or the size of the curve. It has been used, for example, by F. C. Mills in a comparative study of the distribution of prices selected at two different dates. The distribution would not be affected if one commodity price rose from 100% to 200%, while another fell from 200% to 100%. Yet such an internal movement in a distribution of prices may have an economic significance.

To illustrate the process used in the actual calculation of the measures described above, the Bureau of Business Research case is presented. Table 1, following the statement of the case, shows the data as collected for the group of 180 department stores, without any attempt at organization of the data. The calculations and charts which follow are based on the series for gross margin.

As outlined above, the first step is to arrange the figures for gross margin in order of size. This is done in graphical form

in Fig. 27 where it will be noticed that the figures are sorted into groups, each having an interval of 1% between its highest and lowest limits. Furthermore, it will be noticed that the highest and lowest points are so selected that the midpoint of each group has an even 1% for its value. Thus, the modal class includes values from 33.50 up to 34.50. This makes the midpoint equal to 34. The upper portion of the chart representing the distribution is so arranged in graphical form that the height of each column corresponds to the number of stores, which is the same as the number of figures in each column in the lower half of the chart.

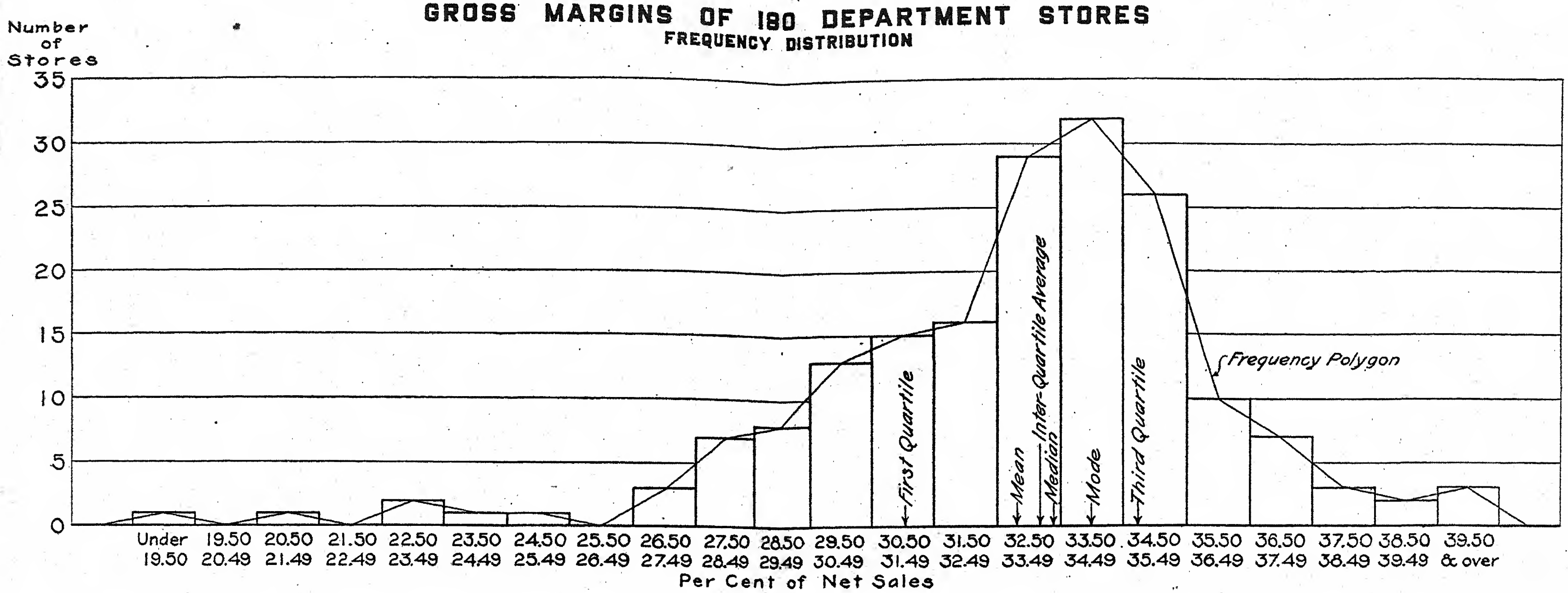
As defined above, a scale value of the point which divides the distribution into exactly two equal parts is the median. It may be referred to as the second quartile. A scale value of the point which cuts off the lowest one-fourth of the numbers, or bisects the first half of the distribution is called the first quartile. A scale value of the point which cuts off the lowest three-fourths of the numbers is called the third or upper quartile.

The interquartile average is the arithmetic average of all those numbers which occur between the first and third quartile.

Table 2 shows the frequency distribution in the form as usually given. Calculations of the average, the dispersion, and the other measures of the frequency distribution, based on the form of distribution as given in Table 2, follow in subsequent tables. It should be noted that the values of the measures of the frequency distribution as calculated from the original figures which are indicated in Fig. 27, and those calculated from Table 2 do not agree exactly. This slight variation is to be expected since the latter calculations were made from the figures grouped in a summary form which assumes that the horizontal scale value of each of the items in a group or class interval may be represented by the scale value of the midpoint of the class interval, and that the items are distributed uniformly within the class interval.

Because there are several measures of dispersion in common use, just as there are several measures of averages, it is worth while to note their meaning. The three most common measures of dispersion are as follows: (1) *average deviation;* an arithmetic mean of all deviations, regardless of sign, of each item from the arithmetic mean of all items, (2) *standard deviation;* a square root of the arithmetic mean of the squared deviations measured from the arithmetic mean of the original items, (3) *coefficient of dispersion* or variation; the standard deviation expressed as a per-

GROSS MARGINS OF 180 DEPARTMENT STORES
FREQUENCY DISTRIBUTION



Symbol

① *First Quartile* = 31.05 (31.048) ----------→ Q_1

② *Median (=Second Quartile)* = 33.39 (33.385) $\left(= \frac{33.38 + 33.39}{2}\right)$ ----→ Me

③ *Third Quartile* = 34.75 (34.748) ----------→ Q_3

④ *Arithmetic Mean* = 32.80 $\left(= \frac{5904.77}{180}\right)$ ----------→ Ma

⑤ *Inter-Quartile Average* = 33.19 $\left(= \frac{2987.18}{90}\right)$

⑥ *Modal Class* = 33.50 to 34.49

⑦ *Mode (approximated)* = 34.00 ----------→ Mo

Inter-Quartile Range

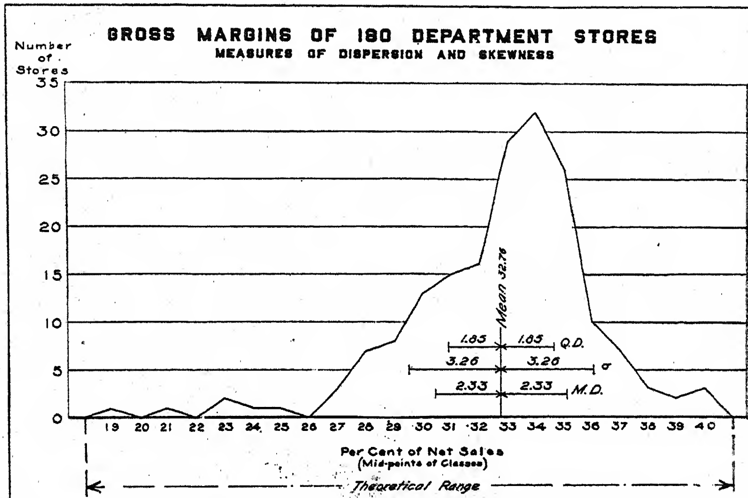
Modal Class

Per Cent of Net Sales	Gross margins (as tabulated, top to bottom)
Under 19.50	13.56
19.50 – 20.49	
20.50 – 21.49	20.70
21.50 – 22.49	
22.50 – 23.49	23.08, 22.92
23.50 – 24.49	24.34
24.50 – 25.49	2468
25.50 – 26.49	
26.50 – 27.49	27.42, 27.20, 27.02
27.50 – 28.49	28.49, 28.45, 28.42, 28.40, 28.33, 27.81, 27.76
28.50 – 29.49	29.39, 29.34, 29.25, 29.21, 29.18, 28.64, 28.63, 28.55
29.50 – 30.49	30.39, 30.33, 30.18, 30.16, 30.14, 30.14, 29.93, 29.90, 29.75, 29.73, 29.70, 29.57, 29.57
30.50 – 31.49	31.36, 31.28, 31.25, 31.19, 31.14, 31.07, 31.07, ① 31.04, 31.00, 30.96, 30.85, 30.80, 30.74, 30.62, 30.58
31.50 – 32.49	32.44, 32.39, 32.33, 32.31, 32.24, 32.16, 32.14, 32.02, 32.01, 31.97, 31.89, 31.87, 31.69, 31.67, 31.61, 31.58
32.50 – 33.49	33.46, 33.44, 33.44, 33.44, 33.41, 33.39, 33.39, 33.38 ②, 33.36, 33.33, 33.31, 33.29, 33.22, 33.18 ⑤, 33.16, 32.93, 32.91, 32.90, 32.90, 32.90, 32.85, ④ 32.82, 32.80, 32.73, 32.69, 32.67, 32.63, 32.56, 32.55
33.50 – 34.49	⑥ 34.46, 34.40, 34.39, 34.38, 34.34, 34.33, 34.32, 34.30, 34.29, 34.29, 34.24, 34.23, 34.22, 34.20, 34.13, 34.08, 34.08, 34.05, 34.04, 34.02, 33.95, 33.87, 33.81, 33.80, 33.76, 33.75, 33.75, 33.73, 33.70, 33.62, 33.59, 33.52
34.50 – 35.49	35.45, 35.37, 35.33, 35.33, 35.31, 35.28, 35.26, 35.20, 35.15, 35.14, 35.13, 35.06, 35.04, 35.03, 34.97, 34.91, 34.82, 34.80, 34.76, 34.75 ③, 34.74, 34.64, 34.64, 34.63, 34.62, 34.57
35.50 – 36.49	36.42, 36.23, 36.15, 36.07, 35.88, 35.83, 35.62, 35.61, 35.60, 35.56
36.50 – 37.49	37.47, 37.41, 37.32, 37.12, 36.89, 36.85, 36.50
37.50 – 38.49	38.45, 37.76, 37.55
38.50 – 39.49	39.38, 38.59
39.50 & over	44.64, 41.89, 41.84

Per Cent of Net Sales

Gross margins are classified according to size, and tabulated, each class as a separate column. The same frequency distribution is shown above the tabulation in the form of a column graph, and also as a frequency polygon.

centage of the arithmetic mean of all items, so that it is a measure which is independent of the units used in a particular distribution. The coefficient of dispersion is valuable when one desires to compare the dispersions of two distributions which are given in different units, or which are widely different in the sizes of their respective means.



Measures of Dispersion — Symbol

Range $= 40.5 - 18.5 = 22\%$

Mean (average) Deviation $= \dfrac{\Sigma d}{N} = \dfrac{419.64 \text{ (total deviations)}}{180 \text{ (number of stores)}} = 2.33\%$ — M. D.

Quartile Deviation $= \dfrac{Q_3 - Q_1}{2} = \dfrac{34.76 - 31.05}{2} = \dfrac{3.71}{2} = 1.65\%$ — Q. D.

Standard Deviation $= \sqrt{\dfrac{\Sigma d^2}{N}} = 3.26$ (per cent of net sales) — σ

Coefficient of Variation $= \dfrac{\sigma}{Ma} \times 100 = \dfrac{3.26}{32.76} \times 100 = 9.95\%$ (per cent of the mean) — V

Measure of Skewness

Skewness $= \dfrac{Q_1 + Q_3 - 2Me}{Q.\,D.} = \dfrac{31.05 + 34.76 - 2(33.28)}{1.65} = -.4054$ — sk

The negative value of sk indicates that the curve is skewed to the left, or toward the smaller items.

FIG. 28.

The calculated values for the measures of dispersion and skewness are shown in Fig. 28. The average deviation of 2.3 means that among the 180 stores the variations of gross margins from the mean figure of 32.76% are on the average 2.3%. The standard deviation 3.26% is somewhat larger because in the process of squaring individual deviations the extreme deviations are always emphasized.

The coefficient of variation makes it possible, for example, to compare the variations in gross margins with those in profits. Since the average profit is a smaller figure than the average gross margin, an equal absolute variation in the profits figure has a greater significance. That fact would be taken into account by comparing the two coefficients of dispersion instead of the absolute deviations.

BUREAU OF BUSINESS RESEARCH

The Bureau of Business Research of the Harvard Business School had collected operating expense statements from 180 department stores with sales of over $1,000,000 for the year 1927. These statements gave figures for net sales, gross margin, profit and loss, and certain expense items such as rent, salaries and wages, and advertising. The figures were collected with the object of analyzing the returns and of finding typical figures for each series so that any given store could compare its position in any particular series with the data for all of the stores. The factor of size of store was eliminated by expressing each item in the statement as a percentage of net sales for that store. The problem of determining the typical or average figure for each series and of studying the distribution of items still remained.

It was known by the Bureau that some trade associations in computing summary figures for the businesses within their own associations simply calculated an arithmetic average of the figures. It was believed, however, that such an average was influenced altogether too much by extreme values. The arithmetic average really should be considered in relation to a frequency distribution, since the arithmetic average is only one of a number of summary figures which might be used. Among the others are the mode and the median, as well as modifications of these and of the arithmetic average. A consideration of the geometric or harmonic means was not included because these seemed to be too complicated for this problem.

The analysis of the frequency distribution of percentage figures for gross margin illustrates the methods used by the Bureau to obtain the desired summary figures. These percentages were grouped by classes as shown in the upper part of Fig. 27. The number of items in each class was assumed to represent the frequency corresponding to the midpoint of the group. This is shown in Table 2. It will be noted that the figures in Table 2, when considered from the point of view of order and size, are arranged in a rough outline which has the bell-shaped characteristic. The outline of the distribution in this case is sufficiently regular so that the modal class could be determined by inspection of the figure. The values of the arithmetic average and median were calculated as shown in Tables 3 and 4.

Because it was desired to compare the median with another figure representative of the distribution, the arithmetic average

of the figures between the upper and lower quartiles was calculated. The values for the upper and lower quartiles and this modified arithmetic average are shown in Table 4. The value 33.1% was chosen as the typical figure for gross margin. Although this value was not the same as any of the calculated measures of central tendency, it was arbitrarily selected because it balanced with the common figures for total expense and net profit.

It was the practice of the Bureau to return to each contributing firm a set of the common figures expressed in terms of percentages of net sales together with the corresponding figures for that particular business. This procedure enabled each store manager to make comparisons which would assist him in the control of his store. It was of interest to the individual store manager to know the significance of the deviation of one of his own figures from the common figure. A knowledge of the significance of this deviation could be obtained only from an understanding of the dispersion in the distribution. The statistician, therefore, calculated the range, as well as other measures of dispersion for each of the distributions. The calculations for the measures of dispersion for gross margin are shown in Tables 5 to 8, and in Fig. 28.

If the distributions were symmetrical, these measures of dispersion would be sufficient for purposes of comparison. Actually, the distributions were seldom symmetrical in form. The amount of skewness, therefore, was important. Although the statistician knew that the meaning and purpose of a measure of the skewness would be difficult to describe in nontechnical language such that the average store manager would understand, a knowledge of the skewness, however, was valuable in giving advice in regard to any particular store. Furthermore, the statistician of the Bureau believed that, in an extended study of facts which the Bureau figures would reveal, measures of skewness necessarily would be included. Therefore, these were calculated as shown in Table 9 and Fig. 28 in order to have available necessary measures when a comprehensive study for a period of years was made.

TABLE I

Operating Figures for 180 Department Stores, 1927
(Net Sales over $1,000,000)
(Figures Expressed as Percentage of Net Sales)

No.	Gross Margin	Total Expense	Net Profit and Loss	Salaries and Wages	Rent	Advertising
1	30.39	31.37	− 0.98	16.07	4.04	2.68
2	34.74	33.05	+ 1.69	18.98	3.61	2.12
3	35.26	33.59	+ 1.67	17.23	2.83	3.09
4	30.74	31.91	− 1.17	19.02	2.95	2.49
5	31.00	29.07	+ 1.93	17.66	1.49	2.00
6	34.80	29.31	+ 5.49	17.02	1.71	2.34
7	33.81	29.67	+ 4.14	16.53	1.86	2.89
8	36.15	31.63	+ 4.52	16.77	3.18	2.72
9	34.04	32.04	+ 2.00	15.43	3.08	4.03
10	31.97	33.27	− 1.30	16.47	3.19	4.87
11	34.08	31.68	+ 2.40	18.12	2.76	3.29
12	39.38	41.19	− 1.81	20.64	4.24	3.22
13	35.88	27.88	+ 8.00	13.30	3.80	3.02
14	35.04	31.21	+ 3.83	16.29	3.04	2.77
15	30.16	32.26	− 2.10	15.72	3.93	3.45
16	34.33	36.24	− 1.91	18.41	1.86	6.77
17	31.87	24.67	+ 7.20	13.09	1.48	2.24
18	24.34	24.69	− 0.35	14.12	2.75	1.99
19	33.75	29.34	+ 4.41	14.60	4.34	3.26
20	33.33	30.68	+ 2.65	17.11	3.16	2.45
21	34.24	29.22	+ 5.02	14.07	2.53	3.28
22	34.22	32.35	+ 1.87	16.43	4.46	3.01
23	32.69	29.08	+ 3.61	16.49	3.03	3.18
24	31.69	36.88	− 5.19	16.30	4.95	6.07
25	33.39	31.22	+ 2.17	15.12	5.37	3.30
26	34.02	36.50	− 2.48	16.11	3.78	5.63
27	34.76	32.57	+ 2.19	17.06	3.70	2.84
28	32.24	33.03	− 0.79	20.14	1.50	4.81
29	41.84	38.63	+ 3.21	20.52	5.04	2.28
30	33.80	30.62	+ 3.18	17.75	2.11	2.75
31	34.05	30.21	+ 3.84	12.80	3.10	2.89
32	29.73	33.50	− 3.77	16.33	2.87	2.99
33	38.59	37.30	+ 1.29	18.89	5.04	3.92
34	35.83	35.06	+ 0.77	16.65	2.68	5.88
35	34.82	31.41	+ 3.41	15.98	3.10	3.38
36	30.14	27.31	+ 2.83	15.02	2.87	3.15
37	32.63	33.60	− 0.97	18.06	2.68	4.27
38	35.03	28.40	+ 6.63	14.90	3.21	2.66
39	33.36	30.95	+ 2.41	15.54	3.16	3.45
40	34.63	29.00	+ 5.63	14.78	4.94	1.41
41	37.12	34.87	+ 2.25	16.36	7.38	2.38
42	34.29	35.41	− 1.12	15.40	5.79	6.01
43	24.68	25.24	− 0.56	10.69	5.40	2.21
44	32.56	29.19	+ 3.37	15.73	4.10	4.03
45	33.16	34.01	− 0.85	16.48	3.57	4.97
46	34.23	35.51	− 1.28	19.98	3.92	4.59
47	34.46	35.55	− 1.09	18.05	3.63	4.17
48	33.76	31.43	+ 2.33	17.48	2.27	3.35
49	29.57	27.67	+ 1.90	16.59	2.67	2.83
50	33.59	29.17	+ 4.42	16.14	3.35	3.09
51	35.14	26.57	+ 8.57	15.34	3.03	2.14
52	29.70	28.07	+ 1.63	14.68	2.18	2.81
53	33.70	33.73	− 0.03	19.67	2.81	1.89
54	29.75	23.54	+ 6.21	12.00	2.96	2.78
55	32.90	31.93	+ 0.97	17.79	2.92	2.21
56	35.37	33.38	+ 1.99	21.32	1.86	2.18
57	32.02	33.05	− 1.03	15.51	5.08	2.29
58	35.28	36.42	− 1.14	19.63	5.19	2.47
59	33.41	28.28	+ 5.13	16.14	2.18	3.00
60	27.02	24.96	+ 2.06	9.68	2.36	2.76
61	35.33	34.21	+ 1.12	17.27	3.21	3.90
62	35.20	31.75	+ 3.45	17.83	4.55	1.84
63	29.21	21.00	+ 8.21	11.26	3.02	1.09
64	31.67	30.65	+ 1.02	13.90	2.80	3.61
65	28.55	29.57	− 1.02	16.88	2.74	1.38
66	30.58	29.13	+ 1.45	16.24	3.06	2.77

TABLE I (*Continued*)

No.	Gross Margin	Total Expense	Net Profit and Loss	Salaries and Wages	Rent	Advertising
67	34.57	31.71	+ 2.86	15.36	3.75	2.56
68	32.85	32.94	− 0.09	15.16	4.76	4.41
69	35.56	34.23	+ 1.33	16.68	4.56	4.33
70	32.14	28.66	+ 3.48	16.01	2.92	2.23
71	35.31	35.65	− 0.34	18.34	2.78	4.19
72	32.73	28.99	+ 3.74	13.44	3.07	2.70
73	28.45	25.55	+ 2.90	12.71	5.45	2.88
74	28.40	28.65	− 0.25	13.77	3.44	3.75
75	32.90	29.13	+ 3.77	13.67	3.08	3.14
76	35.15	31.70	+ 3.45	17.75	4.39	3.43
77	34.39	31.32	+ 3.07	15.51	4.06	2.33
78	31.58	32.30	− 0.72	17.19	2.60	3.14
79	27.42	27.04	+ 0.38	16.27	3.95	0.61
80	37.32	33.05	+ 4.27	14.48	3.31	3.63
81	35.61	34.00	+ 1.61	18.71	2.12	2.70
82	37.76	33.20	+ 4.56	19.53	3.16	2.37
83	34.20	34.48	− 0.28	19.59	4.34	2.53
84	35.45	28.99	+ 6.46	15.26	3.46	2.71
85	30.62	29.15	+ 1.47	14.97	2.56	3.04
86	36.42	31.82	+ 4.60	17.17	4.44	2.07
87	35.13	32.57	+ 2.56	17.49	3.71	3.17
88	32.67	32.25	+ 0.42	16.73	2.51	4.33
89	36.89	33.27	+ 3.62	18.41	4.88	2.38
90	33.73	25.49	+ 8.24	14.65	1.78	2.82
91	32.31	30.52	+ 1.79	13.94	2.65	1.86
92	33.22	27.05	+ 6.17	16.49	2.34	2.17
93	35.06	34.90	+ 0.16	17.66	3.33	3.82
94	30.85	33.13	− 2.28	16.58	6.13	3.41
95	13.56	13.89	− 0.33	6.63	0.32	1.63
96	37.55	33.03	+ 4.52	17.44	3.51	2.99
97	35.60	35.83	− 0.23	17.58	3.74	4.11
98	35.33	35.84	− 0.51	18.00	5.42	2.97
99	37.41	33.32	+ 4.09	16.92	2.73	2.87
100	35.62	29.15	+ 6.47	16.15	2.63	1.95
101	33.52	27.18	+ 6.34	14.52	3.10	4.32
102	34.62	37.97	− 3.35	17.83	3.97	5.11
103	33.44	41.42	− 7.98	24.78	6.95	2.53
104	34.64	33.46	+ 1.18	17.28	5.54	4.50
105	31.04	26.18	+ 4.86	16.36	1.99	3.35
106	28.64	27.47	+ 1.17	13.56	3.56	2.61
107	34.40	30.81	+ 3.59	17.79	0.75	3.89
108	38.45	32.45	+ 6.00	16.18	2.43	2.92
109	36.85	35.11	+ 1.74	20.61	3.24	2.80
110	44.64	33.26	+11.38	15.78	4.10	4.13
111	34.29	31.48	+ 2.81	15.05	3.72	2.75
112	33.31	29.76	+ 3.55	17.17	2.84	2.54
113	27.76	34.78	− 7.02	15.27	4.57	5.40
114	23.08	26.48	− 3.40	12.30	2.85	4.29
115	33.39	33.65	− 0.26	17.58	3.89	4.08
116	32.01	25.79	+ 6.22	14.75	3.51	2.73
117	28.33	34.32	− 5.99	16.27	3.22	5.41
118	33.46	34.78	− 1.32	17.01	4.78	3.31
119	36.23	32.22	+ 4.01	16.62	6.40	2.69
120	32.33	32.90	− 0.67	17.94	2.45	3.39
121	33.44	33.94	− 0.50	17.52	3.66	6.28
122	31.07	30.80	+ 0.27	13.43	1.41	3.47
123	28.49	27.12	+ 1.37	14.25	3.38	2.96
124	22.92	21.18	+ 1.74	11.16	2.55	4.93
125	34.13	29.18	+ 4.95	15.68	2.89	3.65
126	33.44	32.74	+ 0.70	17.76	2.00	3.89
127	30.14	35.65	− 5.51	16.07	3.28	4.44
128	33.16	37.00	− 3.84	19.06	3.74	5.68
129	34.38	34.88	− 0.50	19.09	4.01	3.58
130	32.55	26.62	+ 5.93	15.51	2.77	1.98
131	34.64	30.82	+ 3.82	16.84	3.58	2.97
132	33.75	31.56	+ 2.19	18.48	1.88	3.36
133	32.82	31.98	+ 0.84	18.80	1.44	4.35
134	32.93	29.84	+ 3.09	15.69	2.11	3.44
135	29.57	27.73	+ 1.84	15.84	2.70	1.86
136	32.39	33.75	− 1.36	15.31	4.53	4.63
137	34.30	34.27	+ 0.03	20.46	2.87	3.27

TABLE I (*Continued*)

No.	Gross Margin	Total Expense	Net Profit and Loss	Salaries and Wages	Rent	Advertising
138	28.63	24.97	+ 3.66	12.49	2.31	2.55
139	34.34	30.32	+ 4.02	18.06	2.37	3.72
140	34.91	28.79	+ 6.12	16.66	2.26	2.66
141	34.32	29.92	+ 4.40	16.27	3.56	2.26
142	37.47	31.21	+ 6.26	17.22	2.59	3.23
143	30.80	25.78	+ 5.02	12.10	3.10	2.48
144	20.70	17.75	+ 2.95	7.65	0.99	1.99
145	33.38	32.54	+ 0.84	17.98	2.69	3.64
146	29.93	31.37	− 1.44	16.16	4.25	3.66
147	30.18	28.90	+ 1.28	15.28	3.36	3.24
148	33.62	33.32	+ 0.30	18.88	3.79	3.48
149	30.96	28.17	+ 2.79	15.46	1.75	3.25
150	32.80	29.77	+ 3.03	13.09	5.41	2.85
151	31.36	24.08	+ 7.28	13.05	3.77	2.58
152	27.81	28.00	− 0.19	13.70	4.75	2.36
153	36.07	35.48	+ 0.59	18.69	3.60	2.67
154	29.90	32.77	− 2.87	15.06	3.78	3.49
155	31.07	35.14	− 4.07	13.76	6.69	3.76
156	29.25	29.37	− 0.12	12.08	5.39	3.17
157	30.35	31.84	− 1.49	14.31	4.08	4.22
158	31.89	31.58	+ 0.31	14.70	3.92	4.74
159	29.34	23.60	+ 5.74	12.50	2.87	2.67
160	29.39	32.16	− 2.77	15.98	4.55	3.45
161	32.90	32.15	+ 0.75	14.00	4.93	4.57
162	33.29	30.43	+ 2.86	15.95	2.17	4.67
163	31.19	25.59	+ 5.60	13.42	3.43	2.02
164	32.91	31.85	+ 1.06	16.64	4.18	2.74
165	31.25	33.27	− 2.02	15.55	3.72	3.27
166	34.97	33.45	+ 1.52	17.25	4.95	3.93
167	31.14	30.75	+ 0.39	17.20	2.99	3.08
168	33.87	30.86	+ 3.01	16.31	2.66	3.32
169	31.61	30.23	+ 1.38	13.72	3.04	4.94
170	31.28	36.94	− 5.66	20.47	3.65	4.79
171	32.16	29.31	+ 2.85	16.34	3.38	3.76
172	32.44	29.99	+ 2.45	14.82	5.04	4.00
173	34.75	35.30	− 0.55	15.20	5.47	7.48
174	36.50	29.90	+ 6.60	16.97	2.48	2.74
175	27.20	31.38	− 4.18	14.66	2.27	6.98
176	41.89	40.55	+ 1.34	20.37	4.75	2.83
177	29.18	29.82	− 0.64	15.92	2.48	3.23
178	33.95	33.69	+ 0.62	17.99	1.28	2.74
179	28.42	26.77	+ 1.65	15.54	2.45	1.74
180	34.08	30.43	+ 3.65	16.19	4.64	4.59

Source: Bureau of Business Research, Harvard Business School.

TABLE 2
Frequency Distribution
(Gross Margins of 180 Department Stores)

Class Interval 1%	Frequency f
Less than* 19.50	1
19.50–20.49	0
20.50–21.49	1
21.50–22.49	0
22.50–23.49	2
23.50–24.49	1
24.50–25.49	1
25.50–26.49	0
26.50–27.49	3
27.50–28.49	7
28.50–29.49	8
29.50–30.49	13
30.50–31.49	15
31.50–32.49	16
32.50–33.49	29
33.50–34.49	32
34.50–35.49	26
35.50–36.49	10
36.50–37.49	7
37.50–38.49	3
38.50–39.49	2
39.50 and over	3
	$N = 180$

* As a rule the notation "less than" is not a satisfactory way to indicate the lower limit of the first class interval, because it leaves the interval stretched to an indefinite size. However, since the class in question contains only one item, the exact location of the item has only a limited significance. If a definite class limit was indicated it would be necessary to include four extra classes with zero frequency, because the smallest item is only 13.56%.

Moreover, the size of that item suggests that it is an unusual case so that probably it should not be allowed to have a free influence in the calculation of such measures as mean, range, and average or standard deviations.

Such an arbitrary substitution of the midpoint 19 for that of 14 is, however, open to criticism. It may be conceded that it is more desirable to drop the extreme item entirely or else use it without any arbitrary adjustments if its exact value is known. Then we must bear in mind its influence upon the results when interpreting the significance of calculated measures.

Similar considerations apply to the notation "and over."

In some distributions the exact value of this open end item is unknown. For such cases the item sometimes is assigned arbitrarily to the lowest or highest class. This has been done in the calculations which follow.

TABLE 3

Calculation of Arithmetic Mean, Short-cut Method

(Gross Margins of 180 Department Stores)

Class Interval 1 %	Midpoint m	Frequency f	Deviation in Class Intervals[1] d'	fd'
18.50–19.49	19	1	−14	−14
19.50–20.49	20	0	−13	0
20.50–21.49	21	1	−12	−12
21.50–22.49	22	0	−11	0
22.50–23.49	23	2	−10	−20
23.50–24.49	24	1	− 9	− 9
24.50–25.49	25	1	− 8	− 8
25.50–26.49	26	0	− 7	0
26.50–27.49	27	3	− 6	−18
27.50–28.49	28	7	− 5	−35
28.50–29.49	29	8	− 4	−32
29.50–30.49	30	13	− 3	−39
30.50–31.49	31	15	− 2	−30
31.50–32.49	32	16	− 1	−16
32.50–33.49	Ma' = 33	29	0	= −233
33.50–34.49	34	32	+ 1	+32
34.50–35.49	35	26	+ 2	+52
35.50–36.49	36	10	+ 3	+30
36.50–37.49	37	7	+ 4	+28
37.50–38.49	38	3	+ 5	+15
38.50–39.49	39	2	+ 6	+12
39.50–40.49	40	3	+ 7	+21
		N = 180		= +190

1. Selection of arbitrary origin. Any midpoint may be chosen. It is desirable, however, to take one which seems to be the closest approximation of the value of the mean. Here the Assumed Mean, $Ma' = 33$.

2. Calculation of algebraic sum of deviations $\Sigma fd'$ from arbitrary origin, Ma'

$$\begin{aligned}\Sigma fd' &= -233 + 190 \\ &= -43\end{aligned}$$

3. Calculation of c, correction factor, in class interval units

$$c = \frac{\Sigma fd'}{N}$$

$$\begin{aligned}c &= -\frac{43}{180} \\ &= -0.239 \text{ class intervals}\end{aligned}$$

4. Reduction of c to original units[1]

$$\text{Class interval} = 1\%$$
$$-0.239 \times 1 = -0.239\%$$

5. Determination of Ma

$$\begin{aligned}Ma &= Ma' + c \\ Ma &= 33.0 - 0.239 \\ Ma &= 32.761\%\end{aligned}$$

[1] Since the class interval is equal to 1 %, there is no numerical difference in this instance between deviations expressed in original and in class interval units. Although Step 4 may be omitted in this case, it is essential when the class interval is not equal to the original unit.

TABLE 4

Calculation of Median, Quartiles, and Interquartile Average, Short-cut Method
(Gross Margins of 180 Department Stores)

Calculation of median, *Me*

$$N = 180$$

$$\frac{N}{2} = 90$$

The median is the value at the end of the ninetieth item.

1. Beginning at the top, add the frequency figures in the third column until the sum exceeds 90. Note that addition of 29 brings the sum to 97 which exceeds 90 by 7 items.

2. Subtract 7 from 29. This gives 22. So the ninetieth item is the twenty-second item in the class, 32.50–33.49. The class interval is 1%.

3. Assuming that all 29 values in this class are evenly graduated between the two class limits, the value of the twenty-second item must be more than the lower limit, 32.50, by $\frac{22}{29}$ of the class interval of 1%.

$$Me = 32.50 + \left(\frac{22}{29} \times 1\right)$$

$$= 32.50 + 0.76$$

$$= 33.26\%$$

Calculation of first quartile, Q_1

$$\frac{N}{4} = \frac{180}{4} = 45$$

$$Q_1 \text{ class, } 30.50 - 31.49$$

$$Q_1 = 30.50 + \left(\frac{8}{15} \times 1\right)$$

$$= 30.50 + 0.53$$

$$= 31.03\%$$

Calculation of third quartile, Q_3

$$\frac{3}{4}N = 135$$

$$Q_3 \text{ class} = 34.50 - 35.49$$

$$Q_3 = 34.50 + \left(\frac{6}{26} \times 1\right)$$

$$= 34.50 + 0.23$$

$$= 34.73\%$$

TABLE 4 (*Continued*)

Calculation of Interquartile Average[1]

Class Interval 1%	Midpoint m	Frequency f	Deviation in Class Intervals d'	fd'	
30.50–31.49	31	15	−2	−30	
31.50–32.49	32	16	−1	−16	
32.50–33.49	33	29	0	−46	
33.50–34.49	34	32	+1		+32
34.50–35.49	35	26	+2		52
		N = 118			+84

[1] Theoretically no values below the first quartile or above the third quartile should have been included. In the above computation all items of the classes in which the quartiles fall are included. This makes calculation easier and does not destroy the usefulness of the result.

Calculation

$$\begin{array}{r} +84 \\ -46 \\ \hline \end{array}$$

$$\Sigma fd' = +38$$

$$\frac{\Sigma fd'}{N} = \frac{38}{118} = 0.322$$

$$0.322 \times 1 = 0.322$$

$$\text{Interquartile } Ma = 33 + 0.322 = 33.32\%$$

TABLE 5

Calculation of Range

(Gross Margins of 180 Department Stores)

The range, based on the frequency table is the difference between the highest value that could be included in the last class and the lowest value that could be included in the lowest class. For the upper value we use actually the initial value of the next class interval.

$$40.50 - 18.50 = 22\%$$

TABLE 6

Calculation of Quartile Deviation

(Gross Margins of 180 Department Stores)

Calculation of quartile deviation, based on quartiles as determined by short-cut method

$$QD = \frac{Q_3 - Q_1}{2}$$

$$= \frac{34.73 - 31.03}{2}$$

$$= 1.85\%$$

TABLE 7
Calculation of Mean Deviation, Short-cut Method
(Gross Margins of 180 Department Stores)

Class Interval 1%	Midpoint m	Frequency f	Deviation in Class Intervals d'	fd'
18.50–19.49	19	1	14	14
19.50–20.49	20	0	13	0
20.50–21.49	21	1	12	12
21.50–22.49	22	0	11	0
22.50–23.49	23	2	10	20
23.50–24.49	24	1	9	9
24.50–25.49	25	1	8	8
25.50–26.49	26	0	7	0
26.50–27.49	27	3	6	18
27.50–28.49	28	7	5	35
28.50–29.49	29	8	4	32
29.50–30.49	30	13	3	39
30.50–31.49	31	15	2	30
31.50–32.49	32	16	1	16
		$N_1 =$ 68		
32.50–33.49	$Ma' =$ 33	29	0	0
33.50–34.49	34	32	1	32
34.50–35.49	35	26	2	52
35.50–36.49	36	10	3	30
36.50–37.49	37	7	4	28
37.50–38.49	38	3	5	15
38.50–39.49	39	2	6	12
39.50–40.49	40	3	7	21
		$N_2 =$ 112 $N = (N_1 + N_2)$ $= 180$		$\Sigma fd' = 423$

Calculation of mean deviation, MD

1. Calculation of deviation from arbitrary origin, Ma'
   In class interval units
$$\Sigma fd' = 423$$
   In original units, class interval = 1%
$$423 \times 1 = 423\%$$
2. Calculation of correction factor, c, since deviations were measured from arbitrary origin
   Calculation of d, difference between mean and arbitrary origin
$$d = Ma - Ma'$$
$$Ma = 32.76 \text{ (see Table 3)}$$
$$d = 32.76 - 33 = -0.24$$

Total correction $c = d(N_1 - N_2)$ where N_1 is number of items for which deviation was taken too large (68) and N_2 the number of items for which deviation was taken too small (112).
$$c = -0.24\,(68 - 112) = +10.56\%$$
Since there was a greater number of small deviations than large ones the correction factor is a positive number.

3. Determination of MD
$$MD = \frac{\Sigma fd' + c}{N}$$
$$MD = \frac{423 + 10.56}{180}$$
$$= 2.41\%$$

TABLE 8

Calculation of the Standard Deviation and Coefficient of Variation*
Short-cut Method

(Gross Margins of 180 Department Stores)

Class Interval 1%	Mid-point m	Fre-quency f	Deviation in Class Intervals d'	fd'	$f(d')^2$
18.50–19.49	19	1	−14	−14	196
19.50–20.49	20	0	−13	0	0
20.50–21.49	21	1	−12	−12	144
21.50–22.49	22	0	−11	0	0
22.50–23.49	23	2	−10	−20	200
23.50–24.49	24	1	−9	−9	81
24.50–25.49	25	1	−8	−8	64
25.50–26.49	26	0	−7	0	0
26.50–27.49	27	3	−6	−18	108
27.50–28.49	28	7	−5	−35	175
28.50–29.49	29	8	−4	−32	128
29.50–30.49	30	13	−3	−39	117
30.50–31.49	31	15	−2	−30	60
31.50–32.49	32	16	−1	−16	16
32.50–33.49	33	29	0	0	0
33.50–34.49	34	32	+1	32	32
34.50–35.49	35	26	+2	52	104
35.50–36.49	36	10	+3	30	90
36.50–37.49	37	7	+4	28	112
37.50–38.49	38	3	+5	15	75
38.50–39.49	39	2	+6	12	72
39.50–40.49	40	3	+7	21	147
		$N = 180$			$\Sigma f(d')^2 = 1{,}921$
				−233	
				+190	
				$\Sigma fd' = -43$	

Calculation of standard deviation, σ†

1. Calculation of S^2, the sum of the deviations squared, $\Sigma f(d')^2$, from the arbitrary origin Ma', divided by number of items, N

$$S^2 = \frac{\Sigma f(d')^2}{N}$$

$$S^2 = \frac{1{,}921}{180} = 10.6722 \text{ (in class intervals)}$$

* Compare the first five columns with those of Table 3.

† It could be calculated by the formula, $\sigma = \sqrt{\frac{\Sigma fd^2}{N}}$ if actual deviations, d, were known.

Note that in the short-cut method only assumed deviations, d', are used in the formula, $\sigma^2 = S^2 - c^2$. The derivation of this expression is shown below.

$$d' = d + c$$
$$(d')^2 = d^2 + 2cd + c^2$$
$$\Sigma f(d')^2 = \Sigma fd^2 + 2c\Sigma d + Nc^2$$
$$\frac{\Sigma f(d')^2}{N} = \frac{\Sigma fd^2}{N} + c^2$$
$$\frac{\Sigma f(d)^2}{N} = (\sigma^2) = \frac{\Sigma f(d')^2}{N} - c^2, \text{ where } c = d' - d = \frac{\Sigma fd'}{N}$$

Note: $\Sigma fd = 0$ since d is measured from the arithmetic mean.

TABLE 8 (*Continued*)

2. Calculation of correction factor, since deviations were measured from arbitrary origin

$$c = \frac{\Sigma fd'}{N}$$

$$c = \frac{-233 + 190}{180}$$

$$= -0.2389$$

$$c^2 = 0.0571 \text{ (in class interval units)}$$

3. Calculation of σ

$$\sigma^2 = S^2 - c^2$$

$$\sigma^2 = 10.6722 - 0.0571$$

$$= 10.6151$$

$$\sigma = 3.258 \text{ in class interval units}$$

$$\text{Class interval} = 1\%$$

$$\sigma = 3.258 \times 1$$

$$= 3.258\% \text{ (in original units)}$$

4. Calculation of Coefficient of Variation, V

$$V = \frac{\sigma}{Ma} \times 100$$

$$= \frac{3.26}{32.76} \times 100$$

$$= 9.95\% \text{ (in \% of mean, not in original units)}$$

TABLE 9

Calculation of Measure of Skewness

(Gross Margins of 180 Department Stores)

5. Measure of skewness

$$sk = \frac{(Q_1 + Q_3) - 2Me}{QD}$$

$$= \frac{31.03 + 34.73 - 2(33.26)}{1.85}$$

$$= -0.4108$$

When the value of sk is negative the distribution "tails off" toward the smaller values of the horizontal scale.

CHAPTER III

PROBABILITY PAPER

Probability paper is a graph paper in which the background or grid is so drawn that certain facts about frequency distributions can be determined from the data plotted on it. Although Galton apparently was the first to suggest such a grid for use in problems of social statistics, Hazen was the first to assign the name "probability paper." He used it as a device for applying the theory of probability to some engineering problems on the distribution of rainfall. The value of the grid, however, is not limited to engineering problems, since the business man may use the paper to secure certain controlling facts in regard to problems involving frequency distributions.

Although its name may seem to imply technical difficulties, the use of probability paper is as easy as the use of ordinary plotting paper. The easy graphical method provides the business man with a simple but powerful tool by the use of which he can obtain pertinent information. To obtain this information by numerical methods would require in many cases complicated calculations.

A simple example will show the method of using probability paper. Suppose that our distribution is given as in the following table:

Class Interval	Frequency	Frequency Cumulated	Per Cent of Total
0– 3.9	11	11	3.4
4– 7.9	48	59	18.3
8–11.9	101	160	50.0
12–15.9	101	261	81.7
16–19.9	48	309	96.6
20–23.9	11	320	100.0

The first and second columns present the data. The third column shows the cumulation of the frequencies. We shall see presently

that it makes little difference at which end of the distribution we begin to cumulate. The difference lies only in the interpretation of the result. The fourth column shows the per cent of each cumulated value to the total value. These percentage figures, as shown in the last column, may be plotted on an ordinary

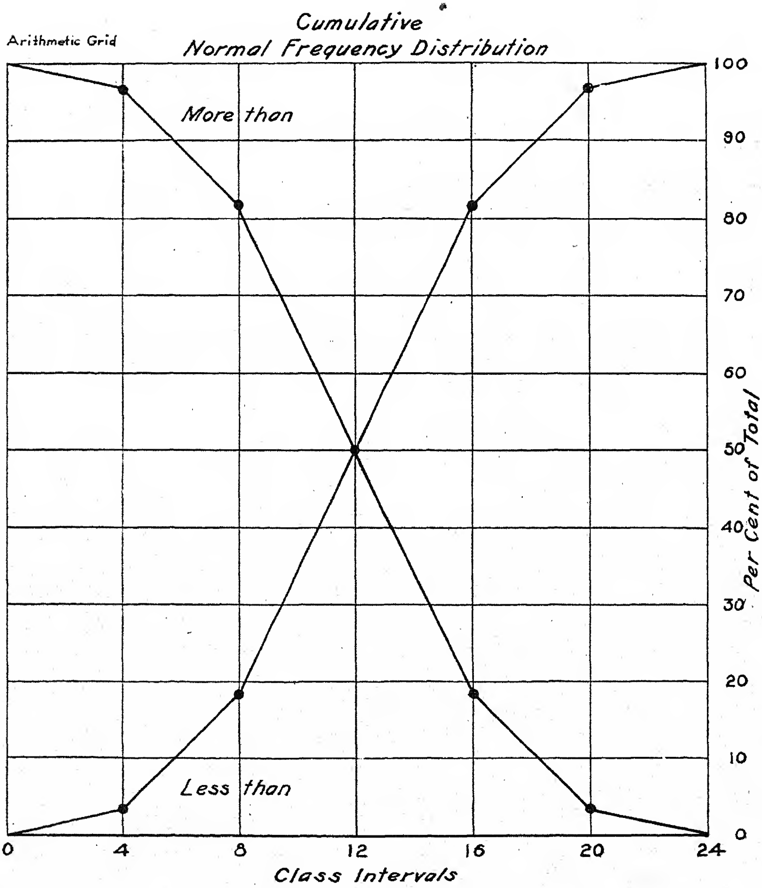


FIG. 29.

arithmetic grid, each figure being plotted on the line corresponding to the upper limit of the class interval, since it represents the cumulative frequencies up to that point. It will be noticed that this curve rises slowly at first, then steeply, and then slowly again as in Fig. 29. This type of curve is called the ogive.

Probability paper is a grid on which the distances between the horizontal lines have been distorted in such a way that when the cumulated percentage frequencies of a normal probability curve are plotted, they will lie on a straight line.

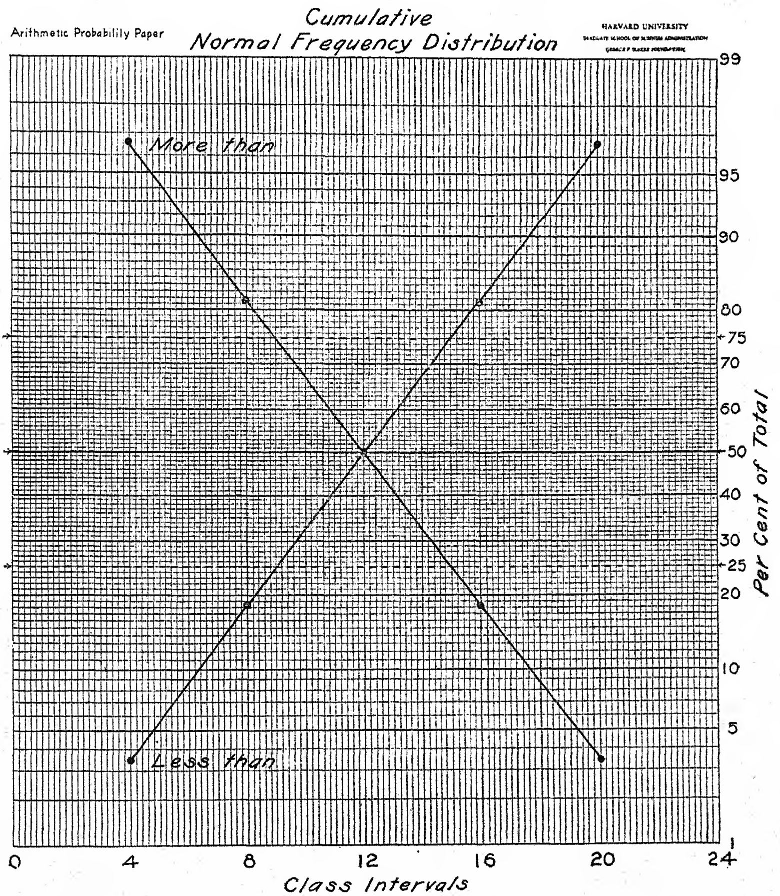


FIG. 30.

Thus, the probability grid conveniently transforms a cumulative frequency curve into a straight line, just as a semilogarithmic grid transforms a compound interest curve into a straight line. Fig. 30 shows the data plotted on a probability grid.

It will be noted at once that the paper has a multiplicity of grid lines drawn upon it. This is apparently at variance with

the principle laid down in the first chapter of the book on graphics. There are, however, two reasons for this exception. The first is that the grid is used as a working device to determine certain values and is not used exclusively as a picture. The second reason is that the horizontal lines are not equally spaced but grow farther and farther apart as we approach the top or the bottom of the grid. A probability scale on the statistician's rule would obviate the necessity of so many grid lines but as one is not available the grid lines are drawn in order to facilitate the location of intermediate percentages.

If the cumulated frequencies in percentage form are plotted, it will be noted that the points lie along a straight line. It is essential to note that the values should be plotted on the line representing the lowest value of the next higher class interval. Thus, in our illustration, 3.4% will be plotted to correspond to 4; 18.3% will be plotted to correspond to 8, and so on. This means that 3.4% of the items are less than 4 and 18.3% are less than 8. A cumulated frequency distribution plotted in this form consequently represents a so-called "less than" curve. Since the spread between the horizontal lines was chosen so that a so-called normal curve (described above) would be straightened out, the plotted points indicate that the example given represents a normal frequency distribution. A distribution which is not normal will plot as a curved line.

It will be noted that we began with the lowest class when we cumulated the frequencies. We might just as well have begun to cumulate the frequencies with the highest class. A table of these cumulated frequencies for the given example is as follows:

Class Interval	Frequency	Frequency Cumulated	Per Cent of Total
0– 3.9	11	320	100.0
4– 7.9	48	309	96.6
8–11.9	101	261	81.7
12–15.9	101	160	50.0
16–19.9	48	59	18.3
20–23.9	11	11	3.4

In plotting the points in this case each plotted point must be on the line which corresponds to the *lower* limit of the class interval. Thus, 3.4% will be plotted against the value 20. This means

that 3.4% of the total frequencies are more than 20. Similarly, 18.3% will be plotted against the value 16, and so on. This curve is called a "more than" curve. The reason is that each percentage represents the percentage of items more than the value against which it is plotted. Comparing the "less than" with the "more than" curve it will be noted that the sum of the percentages of the "less than" and the "more than" curve along any one vertical line is equal to 100%. This means that 100% of the items are both more than and less than certain given values.

A second, somewhat more involved example, together with the determination of certain measures of the distribution, now will be presented.

The frequency distribution of the gross margins of 180 department stores is used as an illustration (see Bureau of Business Research case, page 104). The number of firms was cumulated to show successive totals of firms with gross margins less than the upper limit of each class interval. For example, there was one firm with a gross margin less than 19.50% of net sales; there were 16 firms with a gross margin less than 28.50%. All the reporting firms showed gross margins less than 40.50%. These figures are shown in column 4 of Table 10. Column 5 shows these figures expressed as percentages of the total number of reporting firms. For example, 0.6% of the firms had a gross margin less than 19.50%; 8.9% had less than 28.50%; and 100% were less than 40.50%.

Similarly those firms which had a gross margin of more than a given percentage, were cumulated, as shown in column 4, Table 11. For example, 180 firms or 100% of the distribution showed gross margins of more than 18.50% of net sales; 171 firms or 95% were more than 27.50%; and only three firms or 1.7% more than 39.50%. These cumulative percentages were plotted on probability paper as shown in Fig. 31. The first series constitutes the "less than" curve beginning in the lower left-hand corner and rising as it progresses. The second series constitute the "more than" curve, starting in the upper left-hand corner and descending toward the right. The values on the y-axis using the right-hand scale represent percentages of the total number of stores reporting. The values on the x-axis represent varying percentage of gross margin. The vertical lines represent the limits of the class intervals of the frequency distribution. Values are plotted on the lines rather than in the center of the spaces,

because the data are expressed in cumulative totals up to the class limits. For example, the third plotted point on the "less than" curve shows that 2.2% of the firms have a gross margin less than 23.50%. The third plotted point on the "more than" curve shows that approximately 97.8% of the firms have gross margins greater than 23.50%. The two curves intersect at the 50% line, showing that 50% of the items are less than 33.2 and 50% of the items are more than 33.2. In practice it is necessary to draw only one of the curves.

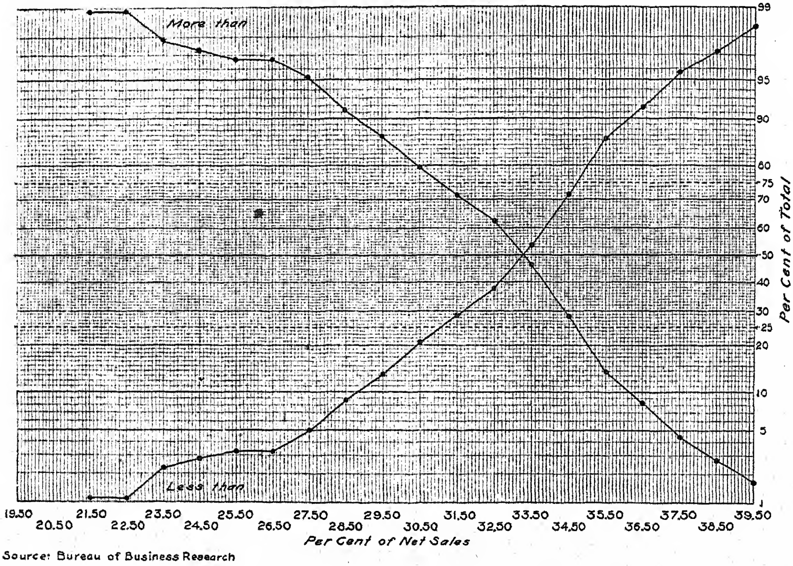
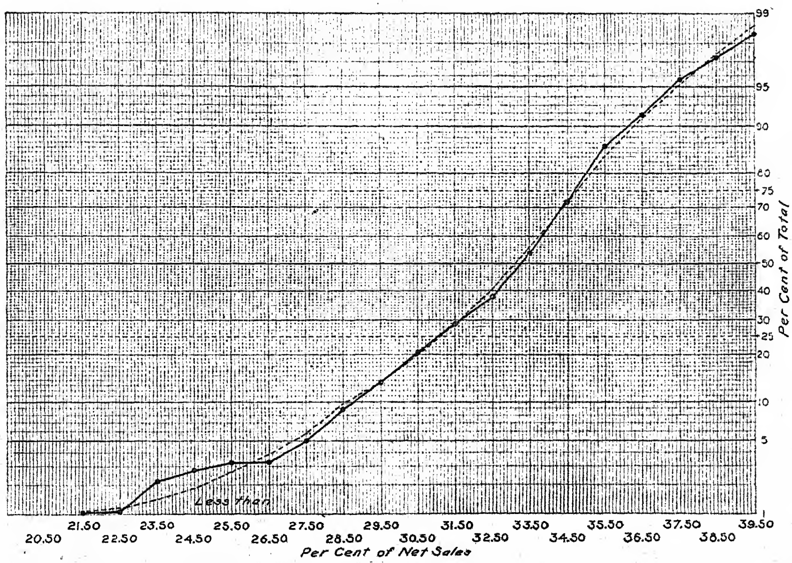


Fig. 31.

The plotted points of the cumulative frequency distribution shown in Fig. 31 do not fall along a straight line. This fact, together with the fact that the shape of the curve below the 50% line differs from the shape of that above, shows that the distribution is neither normal nor symmetrical. The jagged appearance of the curves at the ends is the result of the inadequate sample of items in the end classes. A more representative curve may be obtained by putting a smooth curve through the plotted points by observation as shown in Fig. 32. The readings from this curve are given in Table 12, and a graph of the fitted curve in Fig. 33. If the distribution is close to normal, a normal curve may be

fitted to the data by drawing a straight line through the plotted points. This is illustrated in the case of H. E. Mann, Incorporated, on page 124. The frequencies in the smoothed distribution for the gross margin distribution may be obtained by reading points on the per cent scale corresponding to the points where the fitted curve intersects the vertical lines. The percentages read from the curve are shown in column 2, Table 12. For instance, the readings from the smoothed curve show that 1.1%

GROSS MARGINS OF 180 DEPARTMENT STORES



Source: *Bureau of Business Research*

FIG. 32.

of the firms should have gross margins less than 21.5; that 1.5% of the firms should have gross margins less than 22.5; that 2.2% should have gross margins less than 24.5, and so on. These readings give cumulative percentages, corresponding to the cumulative percentages plotted. Noncumulative percentages may be obtained by subtracting each cumulative figure from the following one. The resulting frequencies in percentage form are shown in column 4, Table 12. These values correspond to the midpoints of their respective classes. For example, 8.3% of the firms are included in the class interval 30.5% to 31.5% which

gives the value centered at the midpoint, 31%. Percentage figures are converted into actual figures by multiplying by the total, 180. The frequencies for the smoothed distribution are shown in column 5. The irregularities which appear in Fig. 33 are caused by the small size of the scale used in the original drawing for Fig. 32.

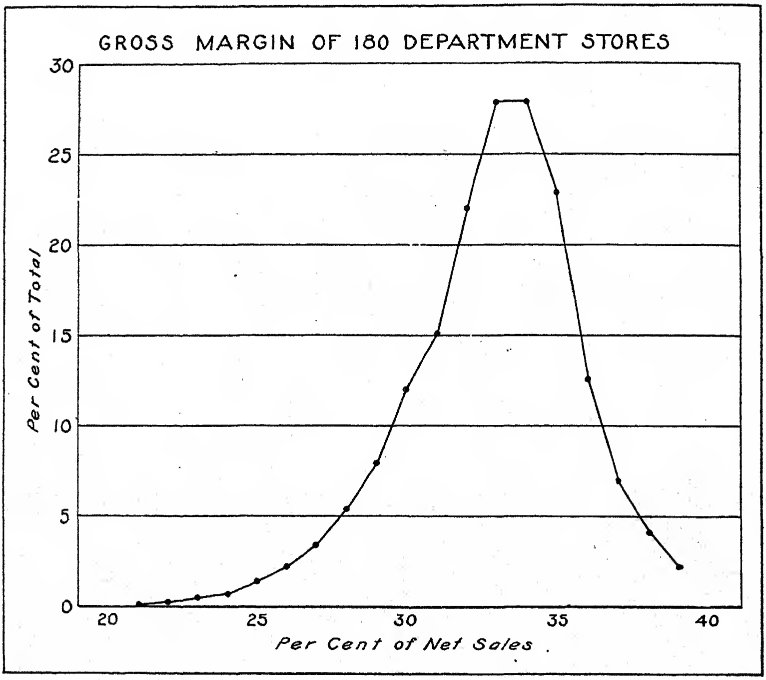


FIG. 33.

When a frequency distribution is presented on probability paper, a complete picture of the distribution is obtained. Many of its characteristics may be determined by reading values from the graph. The median value may be estimated by dropping a line from the point of intersection of the two curves (or from the 50% line, if only one curve is plotted) to the base line, and reading the corresponding value. The value which marks off one half of the items is thus the median value. This value, as noted above is 33.2% (see Fig. 32). Similarly the quartiles may be obtained by reading the values corresponding to the 25% and 75% lines. The values for the quartiles in the gross margin

distribution are 31.1 and 34.7. The deciles may be found in the same manner by the ordinates at the frequencies 10%, 20%, 30%, and so on.

The range between the quartiles will be indicated on the value scale by the corresponding ordinates. Measurement of the distance from the median to the quartiles will give the information necessary to calculate the measure of skewness from the formula $Sk = \dfrac{Q_1 + Q_3 - 2Me}{\dfrac{Q_3 - Q_1}{2}}$. The measurement of skewness in this case is -0.3389. This may be compared with the calculated measure of skewness of -0.4108 as given in Table 9, page 110. The difference is caused partly by the smoothing of the curve and partly by the difficulty of taking accurate readings from the probability paper.

To the business man the measurement of skewness is not so important as a qualitative idea as to its direction. The reason is that skewness is commonly used to modify some central value such as the median. It is wise, however, to estimate the amount of skewness in order to become familiar with the idea.

The arithmetic mean cannot be determined from the graph except when the line is either straight or symmetrical, for then the mean is the same as the median. The relative position of the mean value with respect to the median may be seen readily in connection with the direction of skewness. The latter may be visually approximated from the direction of the line. If the "less than" curve is concave downward, that is, if the lower half of the curve is steeper than the upper half, the mean has a larger scale value than the median because the steepness in the lower values signifies a concentration on the lower values on the left-hand side of the bell-shaped distribution, while the leveling off in the upper half indicates the presence of a "tail" extending toward the higher values. The mean value is distorted by the extreme items and its position to the right of the median shows that in this case half of the total frequency lies below the mean.

When the "less than" cumulative curve is concave upward, it rises more steeply beyond the median, and indicates a concentration on higher values, with a tail to the left. The mean, therefore, is to the left of the median, and is the smaller of the two.

An approximate measure of the standard deviation may be calculated from readings from the graph by the formula

$\sigma = \frac{3}{2}\frac{Q_3 - Q_1}{2}$. This measure is valid only in an approximately normal distribution. Compare the value of the standard deviation of the gross margin series as estimated from curve readings, 2.7 with the computed value of 3.2, Table 8, Bureau of Business Research case, page 110.

Logarithmic probability paper also is known. The grid on this paper has a logarithmic scale for the horizontal or X values. This arrangement tends to flatten a concave downward "less than" curve, because the logarithmic scale widens the spaces representing small values, and condenses those in high values. Thus, a distribution with a peak at the left and a tail at the right will approach more nearly a straight line.

TABLE 10

Cumulated Frequencies

(Gross Margins of 180 Department Stores)

"Less than" Cumulation

Class Intervals (1)	Frequency (2)	Limit of Class Intervals (3)	Cumulative Frequency (4)	Cumulative Frequency, Per Cent (5)
		18.50	0	0.0
18.50–19.49	1	19.50	1	0.6
19.50–20.49	0	20.50	1	0.6
20.50–21.49	1	21.50	2	1.1
21.50–22.49	0	22.50	2	1.1
22.50–23.49	2	23.50	4	2.2
23.50–24.49	1	24.50	5	2.8
24.50–25.49	1	25.50	6	3.3
25.50–26.49	0	26.50	6	3.3
26.50–27.49	3	27.50	9	5.0
27.50–28.49	7	28.50	16	8.9
28.50–29.49	8	29.50	24	13.3
29.50–30.49	13	30.50	37	20.6
30.50–31.49	15	31.50	52	28.9
31.50–32.49	16	32.50	68	37.8
32.50–33.49	29	33.50	97	53.9
33.50–34.49	32	34.50	129	71.7
34.50–35.49	26	35.50	155	86.1
35.50–36.49	10	36.50	165	91.7
36.50–37.49	7	37.50	172	95.6
37.50–38.49	3	38.50	175	97.2
38.50–39.49	2	39.50	177	98.3
39.50–40.49	3	40.50	180	100.0
	$N = 180$			

Source: Bureau of Business Research.

TABLE 11

Cumulated Frequencies
(Gross Margins of 180 Department Stores)
"More than" Cumulation

Class Intervals (1)	Frequency (2)	Limit of Class Intervals (3)	Cumulative Frequency (4)	Cumulative Frequency, Per Cent (5)
		18.50	180	100.0
18.50–19.49	1	19.50	179	99.4
19.50–20.49	0	20.50	179	99.4
20.50–21.49	1	21.50	178	98.9
21.50–22.49	0	22.50	178	98.9
22.50–23.49	2	23.50	176	97.8
23.50–24.49	1	24.50	175	97.2
24.50–25.49	1	25.50	174	96.7
25.50–26.49	0	26.50	174	96.7
26.50–27.49	3	27.50	171	95.0
27.50–28.49	7	28.50	164	91.1
28.50–29.49	8	29.50	156	86.7
29.50–30.49	13	30.50	143	79.4
30.50–31.49	15	31.50	128	71.1
31.50–32.49	16	32.50	112	62.2
32.50–33.49	29	33.50	83	46.1
33.50–34.49	32	34.50	51	28.3
34.50–35.49	26	35.50	25	13.9
35.50–36.49	10	36.50	15	8.3
36.50–37.49	7	37.50	8	4.4
37.50–38.49	3	38.50	5	2.8
38.50–39.49	2	39.50	3	1.7
39.50–40.49	3	40.50	0	0.0
	$N = 180$			

Source: Bureau of Business Research.

TABLE 12

Frequencies from Graphically Fitted Curve
(Gross Margins of 180 Department Stores)

Upper Limit of Class Intervals (1)	Cumulative Percentage Read from Smoothed Curve (2)	Midpoint of Class Intervals (3)	Percentage Differences Centered Opposite Midpoint of Class Intervals (4)	Frequency (5)
18.50	....	.....	....	.....
19.50	....	19.00	....	.....
20.50	....	20.00	....	.....
21.50	1.1	21.00	....	0.18
22.50	1.5	22.00	0.4	0.72
23.50	1.8	23.00	0.3	0.54
24.50	2.2	24.00	0.4	0.72
25.50	2.7	25.00	0.5	0.90
26.50	3.5	26.00	0.8	1.44
27.50	5.5	27.00	2.0	3.60
28.50	8.8	28.00	3.3	5.94
29.50	13.3	29.00	4.5	8.10
30.50	20.6	30.00	7.3	13.14
31.50	28.9	31.00	8.3	14.94
32.50	39.0	32.00	11.1	19.98
33.50	53.9	33.00	14.9	26.82
34.50	71.7	34.00	17.8	32.04
35.50	86.2	35.00	14.5	26.10
36.50	91.7	36.00	15.5	9.90
37.50	95.6	37.00	3.9	7.02
38.50	97.3	38.00	1.7	3.06
39.50	98.4	39.00	1.1	1.98
40.50	100.0	40.00	1.6	2.88

Measures of the distribution, calculated from readings on Fig. 32

$$Me = 33.195 \quad sk = \frac{(Q_1 + Q_3) - 2Me}{\frac{Q_3 - Q_1}{2}} = \frac{65.78 - 66.39}{1.8} = -\frac{0.61}{1.8} = -0.3389$$

$Q_1 = 31.09$
$Q_3 = 34.69$

$$\sigma = \frac{3}{2}\,\frac{Q_3 - Q_1}{2} = \frac{3}{2}(1.8) = 2.7$$

The discussion in the preceding pages has shown how probability paper may be used to obtain figures describing a frequency distribution. Estimates of the characteristics of the distribution of gross margins in the Bureau of Business Research case have been made from probability chart readings. It was pointed out that the somewhat irregular distribution as plotted might be smoothed off by drawing a curve through the data graphically. This process assumes that the irregularities in the measured frequencies are those caused by insufficient observations, so that,

if we had enough observations, a smooth curve would represent the situation more satisfactorily. Drawing a smooth curve on probability paper is equivalent to fitting a smooth curve to the data mathematically. In the case of arithmetic probability paper adjusting a straight line to the data is equivalent to fitting a normal curve to the data. The case of H. E. Mann, Incorporated, will illustrate how this method of curve fitting may be used in a business problem.

H. E. MANN, INCORPORATED

H. E. Mann, Incorporated, was an organization operating a chain of about ninety men's haberdashery stores located in eastern cities of the United States. These stores catered primarily to the large middle class of white-collar office and skilled factory workers.

The executive committee of the parent organization asked one of the subsidiaries, a shirt manufacturing company, to bring out a new line of attractive, light-weight woolen sport shirts appropriate for golf, hunting, fishing, and camping. At a retail price of $3, it was estimated that they should be able to sell 100,000 shirts of this line within the first two months of the fall season.

One problem that called for careful consideration by the executives was the determination of the number of shirts of each size they should order the factory to make. Their previous distribution of sizes had not been entirely satisfactory. Stocks of end sizes accumulated at the factory, since more were manufactured than were ordered by the merchandise managers of the stores. This stock could be disposed of only through special sales at reduced prices.

The assistant factory manager suggested that, if a study were made of the measurements of neck circumferences of a large number of men, it would be comparatively easy to adjust the manufacturing order so that the number of each size ordered would be proportional to the number of customers with corresponding neck measurements. The executives decided to look into this method and requested the statistical department to recommend how they would distribute the proposed lot of 100,000 shirts according to neck sizes.

The available data included a table showing the variation in neck sizes for a large number of men (see columns 1 and 2 of Table 13). The shirt-band sizes as given were standard. These two sets of figures provided the data for the study.

The first step in the solution of the problem was to cumulate the frequencies shown in column 2 of Table 13, and express these "less than" cumulated values as percentages of the total (see column 5). These values were plotted on probability paper as shown in Fig. 34. The plotted points approximated a straight

line. Since it was assumed that the neck sizes should be in the form of a normal distribution, a straight line was fitted to the plotted points, thus smoothing irregularities. The straight line on probability paper gave a normal curve fitted to the distribution of neck sizes.

The next problem was to adjust the shirt sizes to this distribution (see Table 14). Since there was a small difference between the circumference of a man's neck and the length of the shirt band worn, an adjustment was necessary. Some men like their

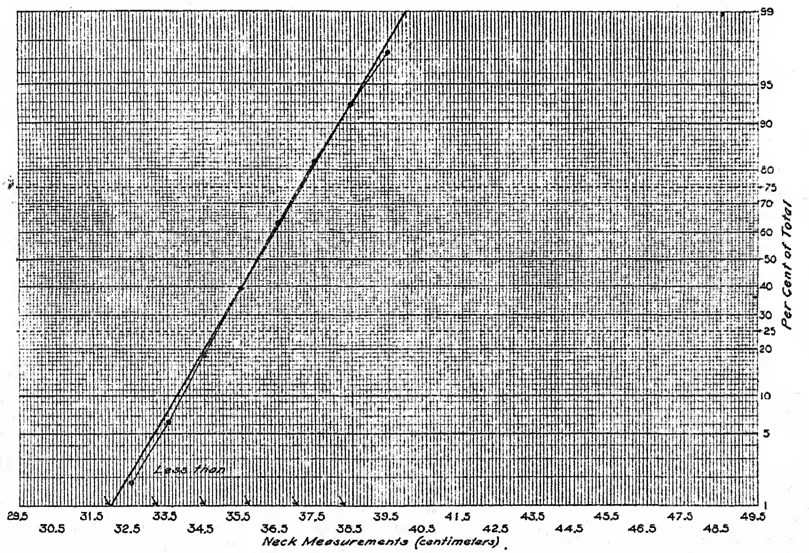


FIG. 34.

collars loose, while others prefer them snug. On the average, experience showed that an allowance of 3 centimeters over and above the neck measurement should be made to secure the proper fit. In other words, a shirt of size 13½ with a band of 34.29 centimeters would fit a man with a neck measurement of approximately 31.29 centimeters. It was assumed that men whose neck measurement varied slightly above and below this figure would be able to wear this size. For instance, men with neck measurements within the limits 30.7 to 31.9 centimeters would be fitted by size 13½. Neck measurements corresponding to different shirt sizes are given in column 4 of Table 14.

The upper limits of the class intervals are indicated in column 2 of Table 15. These values are indicated on the horizontal scale of the graph by interpolating along the scale used for the neck measurements series. Values corresponding to the upper limits of the shirt-size intervals were read from the fitted line, as shown in column 3 of Table 15. These values represented cumulated percentages. To find the percentage belonging to each class group, differences were taken. These were centered opposite the midpoint of the class interval and were considered to represent the number of shirts of that size to be manufactured (see columns 4 and 5, Table 15). Since the scale used on this sheet of probability paper does not give the readings for the extreme values, the remaining 3% were assigned arbitrarily to the beginning and end class intervals. Percentage values were converted into actual values by multiplying by the total number of shirts.

A larger number of digits could have been secured if the graph were on a larger scale or if a mathematical process employing tables had been used to calculate a smooth curve for the distribution. Readings from probability paper are accurate to 0.1 or 0.2 of 1%. From a practical manufacturing or merchandising point of view this degree of accuracy is sufficient so that additional digits are of questionable value.

TABLE 13

Neck Measurements of White Troops at Demobilization

Neck Measurements, Centimeters	Number of Men	Upper Limit of Class Interval	Cumulative Frequency	Cumulative Frequency, Per Cent of Total
28.5–29.49	55	29.5	55	0.06
29.5–30.49	219	30.5	274	0.29
30.5–31.49	314	31.5	588	0.62
31.5–32.49	1,133	32.5	1,721	1.81
32.5–33.49	4,286	33.5	6,007	6.32
33.5–34.49	11,353	34.5	17,360	18.25
34.5–35.49	20,094	35.5	37,454	39.38
35.5–36.49	22,628	36.5	60,082	63.18
36.5–37.49	18,047	37.5	78,129	82.15
37.5–38.49	10,051	38.5	88,180	92.72
38.5–39.49	4,426	39.5	92,606	97.38
39.5–40.49	1,716	40.5	94,322	99.18
40.5–41.49	492	41.5	94,814	99.70
41.5–42.49	147	42.5	94,961	99.85
42.5–43.49	52	43.5	95,013	99.91
43.5–44.49	23	44.5	95,036	99.93
44.5–45.49	22	45.5	95,058	99.95
45.5–46.49	17	46.5	95,075	99.97
46.5–47.49	16	47.5	95,091	99.99
47.5–48.49	5	48.5	95,096	99.99
48.5–49.49	6	49.5	95,102	100.00
	95,102			

Source: *Reports of the Medical Department of the United States Army in the World War*, Vol. 15, Part I, page 538.

TABLE 14

Shirt Sizes

H. E. Mann, Incorporated

Shirt Bands, Inches	Shirt Bands, Centimeters	Shirt-band Length Less 3 Centimeters	Range of Neck Sizes for Given Shirt Sizes, Centimeters
13	33.02	30.02	29.4–30.69
13½	34.29	31.29	30.7–31.89
14	35.56	32.56	31.9–33.19
14½	36.83	33.83	33.2–34.49
15	38.10	35.10	34.5–35.69
15½	39.37	36.37	35.7–36.99
16	40.64	37.64	37.0–38.29
16½	41.91	38.91	38.3–39.49
17	43.18	40.18	39.5–40.79
17½	44.45	41.45	40.8–42.09
18	45.72	42.72	42.1–43.39

TABLE 15

Determination of Number of Shirts

H. E. Mann, Incorporated

Shirt Bands, Inches	Upper Limit of Shirt Band Range Centimeters	Normal Cumulative Frequency Reading, Per Cent of Total*	Normal Non-cumulative Frequency, Per Cent of Total Centered	Number of Shirts Basis, 100,000 Shirts
13			0.2†	200
	30.7			
13½			0.5†	500
	31.9	1.0		
14			4.4	4,400
	33.2	5.4		
14½			13.6	13,600
	34.5	19.0		
15			25.0	25,000
	35.7	44.1		
15½			28.0	28,000
	37.0	72.1		
16			18.5	18,500
	38.3	90.6		
16½			8.4	8,400
	39.5	98.0		
17			1.0†	1,000
	40.8			
17½			0.3†	300
	42.1			
18			0.1†	100
	43.4			
				100,000

* This column has been read from graph.

† Frequencies at extreme ends of the curve cannot be determined from this probability paper. Marked values have been roughly estimated to make the total approach 100%.

CHAPTER IV

INDEX NUMBERS

An index number is a summary number which presents in a single figure a number of facts. Index numbers are used more often with price series than with any other type of economic data. An index number which represents the cost of living is an illustration. We pay different prices for each of the many commodities which we buy. These commodities are part of the necessities or luxuries of our everyday living. It is desirable to know whether these commodities cost us more or less from time to time. Except in times when all prices are advancing radically, it is very difficult to say offhand whether the cost of living at any time is greater or less than at some previous time. Some prices may be higher; while others may be lower. The purpose of an index number in this case is to sum up for any particular date these different prices in order to obtain a single summary number which will represent the cost of living as a whole.

Since relative comparisons can be made better by the use of relative figures than by absolute ones, indexes are usually expressed in percentage form. In a narrower sense used by some writers the definition of an index excludes the possibility of other than the relative form. Because there is no commonly accepted clear-cut definition, the term "index" is often used in a broader sense. The Dunn's and Bradstreet's wholesale price series, for example, are given as summations of individual prices which are not expressed in relative form. Though these series would be excluded by some definitions, they are called "indexes" for the lack of a better name.

An index is built always with the purpose of answering a specific inquiry. The nature and detailed conditions of the inquiry should determine the selection of the series to be included, the method of representing each phenomenon, and the way the component series should be combined. The reverse is also true. An index constructed supposedly for a "general purpose" is really limited to answer an implicitly defined inquiry. An index

of cost of living, for example, indicates the cost only in the particular places for a certain type and size of family having definite spending habits and using particular kinds of products in certain relative amounts.

From what has been said it will be understood that an index is a series of summary (total or average) figures, each of which represents two or more individual values. In an index each of the summary figures is called an "index number." The plural "index numbers" will be used here to indicate several of these summary figures. The word "index" is used when the whole series is referred to without any particular figure or figures being indicated. Thus, we speak of the Index of Production of the Federal Reserve Board. When we speak of the index number for January, 1931, however, it is common practice to speak simply of the index for January, 1931, as there is only one number for that date.

Several related indexes may be so constructed as to allow a "two-way" comparison.[1] For example, a number of indexes have been devised to measure the utilization of machinery in several departments of a factory. The vertical comparison shows the relative changes and the progress which occurred within each department over a period of time. The horizontal, or cross-section comparison indicates the condition of each department in relation to the others, at the same time.

Index numbers have been found very useful in connection with a study of price series, especially wholesale prices. It is more difficult to collect reliable data on retail prices because of their wide variation among different stores. Wholesale prices exhibit greater uniformity in different places at the same time and, therefore, can be obtained more easily. This is because for many commodities there are established wholesale prices which are quoted nationally, and in some cases internationally. Moreover, the wholesale prices, as a rule, change more quickly and fluctuate more widely than the retail prices, in response to changing economic conditions. A wholesale index is thus more sensitive and, therefore, has more significance to those who must watch price variations.

In addition to price indexes there are many indexes which represent other facts for the country as a whole, or for different

[1] See Brown, T. H., *Problems in Business Statistics*, Southern States Textile Mills Case, page 293.

regions. Many of them indicate the general business conditions of the country as reflected in the volume of production, transportation, and sales of certain commodities, in the prices of stocks and bonds, interest rates, and many other series. There are also many indexes which have special significance for individual businesses.

Numerous ways of constructing index numbers have been developed. For our purposes, however, only four will be considered. In order of the simplicity of their construction they are: (1) the sum of the individual values, (2) the arithmetic average of the relative values, (3) the arithmetic average of the weighted relative values, (4) the sum of weighted values divided by the base similarly constructed. Construction of the indexes according to these methods is illustrated in the Sackville Water Company case.

The index of the first type is very simple to construct since it is merely the sum of the actual unit prices. It is frequently called the "aggregate." From a practical point of view, however, this type of index is not generally useful because the prices of the series to be combined may be quoted in different units, such as dollars per ton, cents per pound, dollars per hundred pounds, dollars per bushel, or any other unit form. It is obvious that, if the individual values differ widely, the small values will be lost in the aggregate. Therefore, the time changes will reflect predominantly those of the high values. For example, the price of iron in dollars per ton added to the price of lead in cents per pound gives an index which reflects chiefly the changes in the price of iron. If it is desired that the individual values exercise about the same amount of influence upon the aggregates, it may be helpful to reduce all values to a common basis. For example, individual prices may be expressed in terms of a common unit, such as prices per pound. This device, however, may not bring the commodity prices to a common level, for some commodities may be considerably more expensive than others. For example, if the price of platinum per pound is added to the price of lead per pound, the latter, represented only by a few cents, will be lost again in the aggregate; and the relative changes in the aggregates will be dominated by the changes in the price of platinum.

An analogous difficulty may arise when the prices of some commodities differ even temporarily from the common level. For example, if we added the prices of rubber and cotton, both

expressed in cents per pound, the aggregates would have shown in recent years very uncertain results. Though at times the two prices have been nearly alike, at other times the price of rubber was on much higher levels, and, of course, would have assumed a larger share of influence upon the aggregates. At such times the relative changes in the aggregates would be dominated by the fluctuations in the price of rubber. A diagram illustrating the construction of this type of index is shown in Fig. 35 (see also Table 16 of the Sackville Water Company case).

From the point of view of the mathematical process, the next more complicated index number is the arithmetic average of relative values. By the term "relative value" is indicated

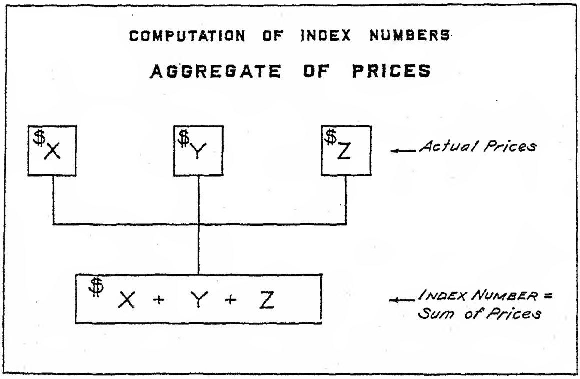


Fig. 35.

the ratio of a value at any time to the corresponding value at a particular time. These ratios usually are multiplied by 100. They are thus expressed in percentage form. Usually, in calculating the relatives of each series, the value taken for the divisor is kept fixed. Whether it is kept fixed or whether it changes in accordance with some plan, it is known as the base value. A very obvious advantage of price relatives is that they reduce to a comparable basis prices representing different units or different levels of value. Thus, in the illustration used above, the relative price of pig iron will be entirely comparable to the relative price of lead or the relative price of copper to the relative price of platinum. Furthermore, such relatives show the changes which occur from time to time in the same series. The main objection to the use of the average of relatives index is that this index does not

reflect the relative importance of the respective series. Figure 36 illustrates the method of construction, and Table 17 of the Sackville Water Company case gives an example of calculation of this type of index.

The third type of index overcomes the above difficulty by weighting the relatives for the various series in accordance with their respective importance.[1] For example, in the Sackville Water Company case a small amount of lead was used together with a large amount of pig iron. The weights represent the rela-

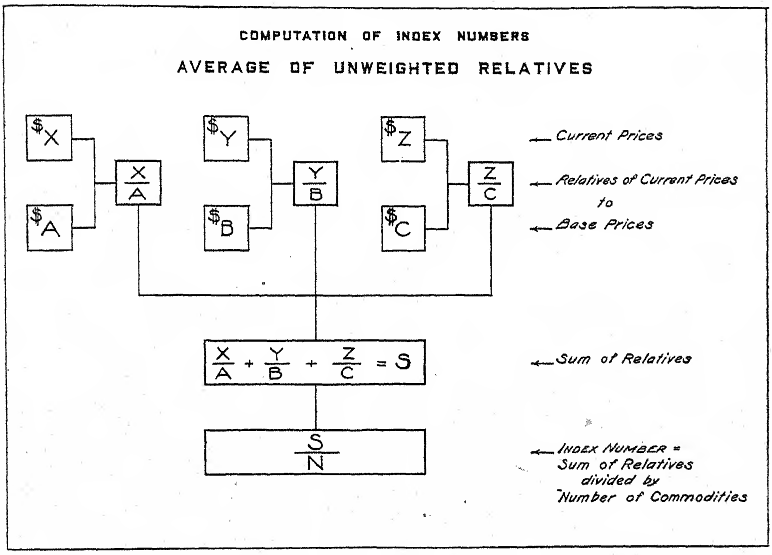


FIG. 36.

tive cost of each commodity to the company. These are obtained by multiplying the price of each commodity in the base year by the typical quantity of each respective commodity used during a year. The typical quantities might be actual figures for the base year or any other representative year, or, as in this case, the average quantities. For simplicity in calculations, the total of the weights is set equal to 100. Therefore, in our problem the weights were obtained by expressing each product of the base price and the respective quantity as percentages of the total value

[1] Though for simplicity the two averages discussed above are often called "unweighted" indexes, they should be more exactly described as "implicitly weighted," because the omission of weighting implies the use of equal weights.

of those products. In the example considered, the relatives for pig iron were multiplied by one weight, the relatives for lead by another and so on. After this was done, the weighted relatives for each month were averaged by dividing the total by the sum of the weights, which is 100. This is the third type of index and is illustrated in Fig. 37, and Table 18 of the Sackville Water Company case.

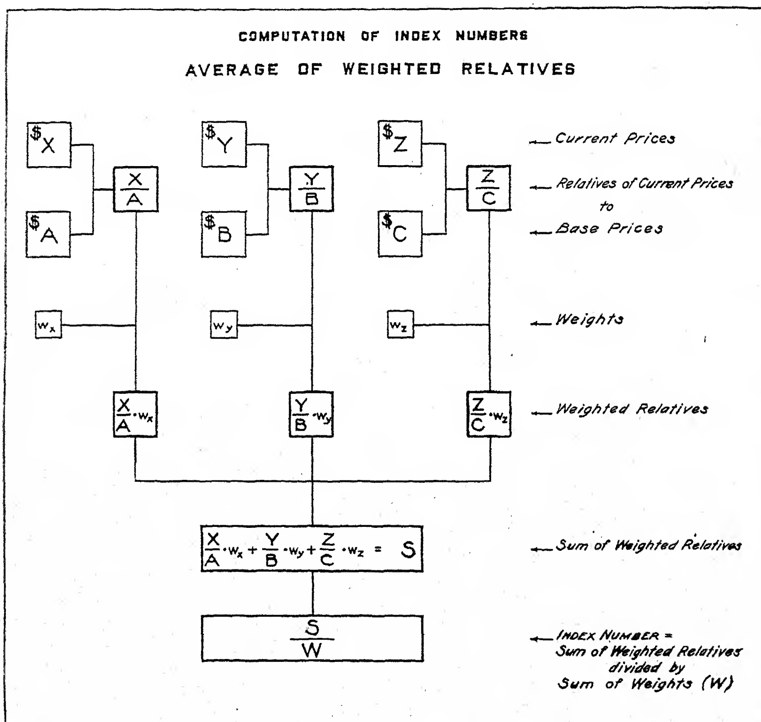


FIG. 37.

In the preceding paragraph a brief statement was made in regard to the way in which the weights should be constructed for a particular index. For purposes of many indexes it is sufficiently accurate to select the weights on the basis of judgment.

In the construction of index numbers showing prices, another type seems to be somewhat more desirable than those already described. This index is constructed as follows: The actual price of each commodity for each period is multiplied by the

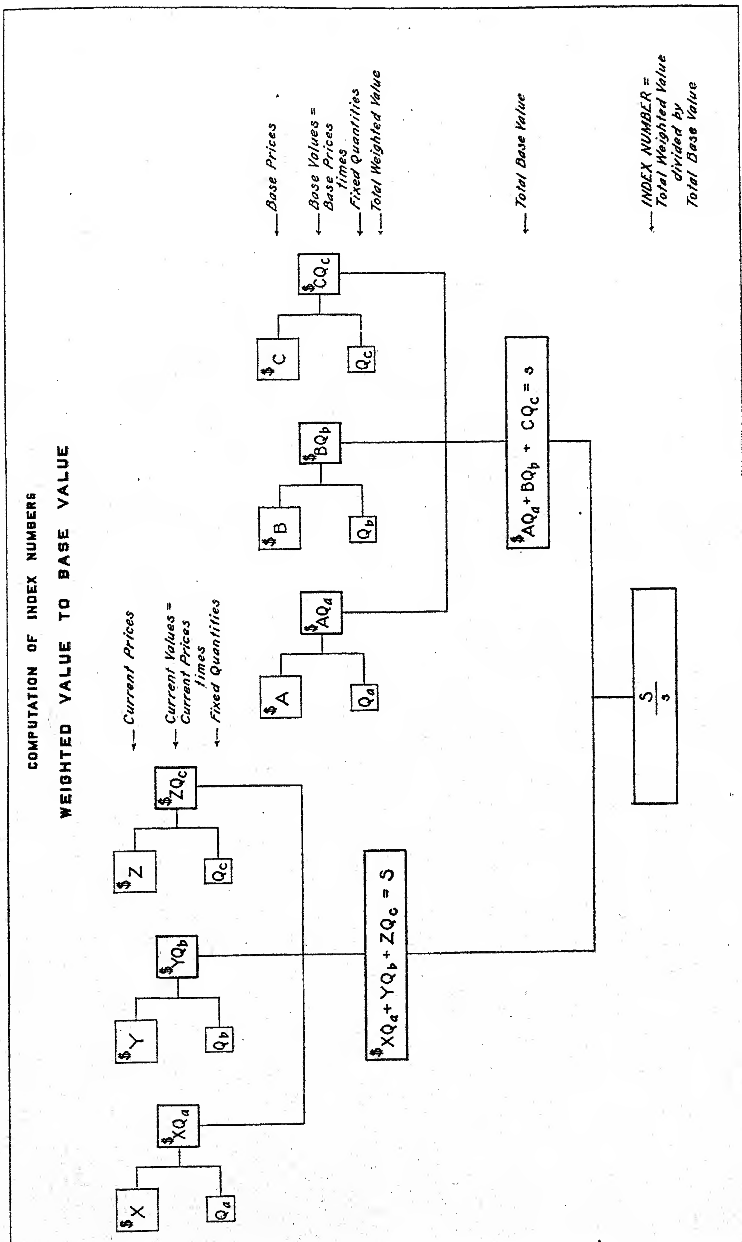

COMPUTATION OF INDEX NUMBERS
WEIGHTED VALUE TO BASE VALUE
Current Prices
Current Values = Current Prices times Fixed Quantities
Base Prices
Base Values = Base Prices times Fixed Quantities
Total Weighted Value
Total Base Value
INDEX NUMBER = Total Weighted Value divided by Total Base Value
$X
$Y
$Z
Qa
Qb
Qc
$XQa
$YQb
$ZQc
$XQa + YQb + ZQc = S
$A
$B
$C
$AQa
$BQb
$CQc
$AQa + BQb + CQc = s
S/s

Fig. 38.

respective typical quantity. Each quantity is selected as in the case above, and is fixed in order to make the index reflect the changes in price only, but not in both the price and the quantity. The sum of the products for each date is then computed. For a base a particular year is chosen and the total weighted value is similarly computed for that year. This is the total base value by which the other total values are divided. The diagram showing the plan of construction of this index is given in Fig. 38. Computations are given in Table 19 of the Sackville Water Company case.

SACKVILLE WATER COMPANY[1]

Use of Index Numbers

The Sackville Water Company supplied a small city in the eastern part of the United States. Although the company had been organized in 1900, it was not until 1905 that its mains had been extended to reach all parts of the community. In 1905 approximately 15 miles of mains had been laid.

In 1927 the directors of the Sackville Water Company adopted the policy of basing depreciation charges on replacement cost. This accounting practice made allowance for the increase in prices of materials and labor since 1905. Such allowance was necessary because the recent growth of the town made it imperative for the company to extend its capacity so that in many cases small pipe had to be replaced by larger pipe. This was expensive. The practice of basing depreciation charges on current rather than historical cost also allowed adjustment of rates to cover changes in costs of materials.

The company accountant believed that replacement cost could be estimated from year to year by the use of an index number which would be simple and easy to construct. In order to choose the best type of index for his purpose he considered four indexes, and compared the resulting values. These indexes, for illustrative purposes, have been constructed on the following pages. The index numbers were based on price quotations for lead, pig iron, and labor, the three most important components in the final cost of the mains. It was estimated that the average weight of the pipe was 80 gross tons per mile, that an average of about 1,500 pounds of lead per mile was used to seal the joints, and that approximately 300 man-days per mile were required to dig and fill the 4-foot ditch.

The first type of index considered by the statistician was an aggregative type, computed by adding the price of pig iron per ton, the price of lead per pound, and the price of common labor per day. The data for the original prices and the computed index are shown in Table 16. *The second type* of index was an unweighted average of relatives. Each item was expressed as a percentage of its June, 1914, value, and an arithmetic average of the three series of relatives was taken. The construction of

[1] *Cf.* Brown, T. H., *Problems in Business Statistics,* Chesapeake and Potomac Telephone Company, page 413.

the index is shown in Table 17. *The third type* was a weighted average of relatives, in which the relatives of each series were multiplied by their respective ratio of the cost of the individual item to total cost per mile of the mains in 1914 as computed from 1914 prices and estimates of average quantities. Finally, the weighted relatives of each series were added together for each year and divided by 100. The method is illustrated in Table 18. *The fourth type* was a weighted aggregate index. The individual prices were multiplied respectively by the average quantity needed to construct one mile of mains, and the totals were taken. Finally, these aggregates were expressed as relatives to the 1914 aggregate value as a base, Table 19.

Which type of index was the best one to use?

TABLE 16

Price Index of Materials and Labor
Unweighted Aggregative Index
Sackville Water Company

Year	Lead, Dollars per Pound	Composite Pig Iron, Dollars per Ton	Labor,[1] Dollars per Week	Index, Dollars
1913	0.0426	14.68	.....	.....
1914	0.0374	12.87	12.45	25.36
1915	0.0457	13.54	12.83	26.42
1916	0.0680	19.43	14.48	33.98
1917	0.0892	36.11	16.38	52.58
1918	0.0725	33.24	20.32	53.63
1919	0.0555	28.97	23.50	52.53
1920	0.0793	42.76	28.19	71.03
1921	0.0439	22.58	25.78	48.40
1922	0.0552	24.06	25.02	49.14
1923	0.0735	26.30	27.18	53.55
1924	0.0810	20.89	27.69	48.66
1925	0.0892	20.59	28.32	49.00
1926	0.0825	20.26	28.96	49.30
1927	0.0652	18.56	29.34	47.97

[1] Labor data begin with 1914.
Source: *Standard Trade and Securities Service.*

TABLE 17

Price Index of Materials and Labor
Unweighted Average of Relatives
Sackville Water Company
(June, 1914 = 100)

Year	Price Relatives			Average of Relatives
	Lead	Pig Iron	Labor	
1914	98	99	98	98.3
1915	120	104	101	108.3
1916	179	150	114	147.6
1917	235	278	129	214.0
1918	191	256	160	202.3
1919	146	223	185	184.7
1920	209	329	222	253.3
1921	116	174	203	164.3
1922	145	186	197	176.0
1923	193	203	214	203.3
1924	213	161	218	197.3
1925	235	159	223	205.7
1926	217	156	228	200.3
1927	172	143	231	182.0

Prices in base period, June, 1914
Lead $0.0380 per pound
Pig iron $12.97 per gross ton
Labor $12.70 per week
Source: *Standard Trade and Securities Service.*

TABLE 18

Price Index of Materials and Labor

Weighted Average of Relatives

Sackville Water Company

(June, 1914 = 100)

Determination of weights

Prices in base period (June, 1914)
Lead \$0.038 per pound
Pig iron \$12.97 per ton
Labor \$12.70 per week

Average quantity
Lead 1,500 pounds
Pig iron 79 tons
Labor 300 man-days

Calculation of total cost

Sum of products of prices in base period and average quantities

$$\underset{\text{Lead}}{(1{,}500 \times \$0.038)} + \underset{\text{Pig Iron}}{(79 \times \$12.97)} + \underset{\text{Labor}}{\left(300 \times \frac{\$12.70}{6}\right)}$$

$$\$57.00 + \$1{,}024.63 + \$635.00 = \$1{,}716.63$$

Cost of components, as percentages of total cost

	Lead	Pig Iron	Labor
	3.3%	59.7%	36.9%
Weights	3%	60%	37%

Calculation of the Index

Year	Lead		Pig Iron		Labor		Weighted Index
	Price Relative	Weighted 3%	Price Relative	Weighted 60%	Price Relative	Weighted 37%	
1914	98	2.94	99	59.40	98	36.26	98.6
1915	120	3.60	104	62.40	101	37.37	103.4
1916	179	5.37	150	90.00	114	42.18	137.6
1917	235	7.05	278	166.80	129	47.73	221.6
1918	191	5.73	256	153.60	160	59.20	218.5
1919	146	4.38	223	133.80	185	68.45	206.6
1920	209	6.27	329	197.40	222	82.14	285.8
1921	116	3.48	174	104.40	203	75.11	183.0
1922	145	4.35	186	111.60	197	72.89	188.8
1923	193	5.79	203	121.80	214	79.18	206.8
1924	213	6.39	161	96.60	218	80.66	183.7
1925	235	7.05	159	95.40	223	82.51	185.0
1926	217	6.51	156	93.60	228	84.36	184.5
1927	172	5.16	143	85.80	231	85.47	176.4

Source: *Standard Trade and Securities Service.*

TABLE 19

Price Index of Materials and Labor[1]

Weighted Aggregative Index

Sackville Water Company

Calculation of Index

Year	Lead		Pig Iron		Labor		Weighted Aggregative Index	Index Relative to 1914 as a Base
	Price	Price × Quantity	Price	Price × Quantity	Price	Price × Quantity		
1913	0.0426	63.90	14.68	1,159.72	.....	....	....	.....
1914	0.0374	56.10	12.87	1,016.73	12.45	622.50	1,695.33	100.0
1915	0.0457	68.55	13.54	1,069.66	12.83	641.50	1,779.71	105.0
1916	0.0680	102.00	19.43	1,534.97	14.48	724.00	2,360.97	139.3
1917	0.0892	133.80	36.11	2,852.69	16.38	819.00	3,805.49	224.5
1918	0.0725	108.75	33.24	2,625.96	20.32	1,016.00	3,750.71	221.2
1919	0.0555	83.25	28.97	2,288.63	23.50	1,175.00	3,546.88	209.2
1920	0.0793	118.95	42.76	3,378.04	28.19	1,409.50	4,906.49	289.4
1921	0.0439	65.85	22.58	1,783.82	25.78	1,289.00	3,138.67	185.1
1922	0.0552	82.80	24.06	1,900.74	25.02	1,251.00	3,234.54	190.8
1923	0.0735	110.25	26.30	2,077.70	27.18	1,359.00	3,546.95	209.2
1924	0.0810	121.50	20.89	1,650.31	27.69	1,384.50	3,156.31	186.2
1925	0.0892	133.80	20.59	1,626.61	28.32	1,416.00	3,176.41	187.4
1926	0.0825	123.75	20.26	1,600.54	28.96	1,448.00	3,172.29	187.1
1927	0.0652	97.80	18.56	1,466.24	29.34	1,467.00	3,031.04	178.8

[1] Quantities, from Table 18
Lead 1,500
Pig iron 79
Labor 50
Source: *Standard Trade and Securities Service.*

CHAPTER V

TIME SERIES ANALYSIS

The reader of business history realizes again and again, in connection with everyday affairs, the effect of the changes which occur with the passing of time. These changes appear in figures which picture quantitatively the facts of business history. Whether the figures represent one enterprise or a group of similar enterprises, there always seems to be stamped upon them the march of progress, the changing mood of the seasons of the year, the waxing and waning of the economic activity of men, and the effects of fortune, whether good or bad. Although these things are understood and recognized by the thoughtful individual, the practical skill of measuring them and of weighing one against another belongs to the kingdom of knowledge of the statistician.

The wise executive not only tries to discover his mistakes from the facts of the past, but also attempts to foresee the future. The attempt to divine the future is not limited to stock market enthusiasts. The difference between the business man and the idle speculator, who knows only the jargon of the stock market, is that the business man has at his command a store of facts. Because they are often complex, the facts must be organized and analyzed for the solution of a given problem. Since economic and social environment changes but slowly, an analysis of series representing past events ought to give some of the factors needed to estimate roughly the events of the future. In a practical case an estimate of the future sales of a company or of the total production for a group of similar businesses might be based on data representing the past.

No matter whether the analysis of the facts of the past are made to discover mistakes or to divine the future, four elements in the quantitative analysis are recognized. These are technically known as (1) long time or secular trend, (2) seasonal variation, (3) cyclical fluctuations, and (4) accidental changes.

The method, called time series analysis, generally implies one of two different processes. The first is an analytical process by

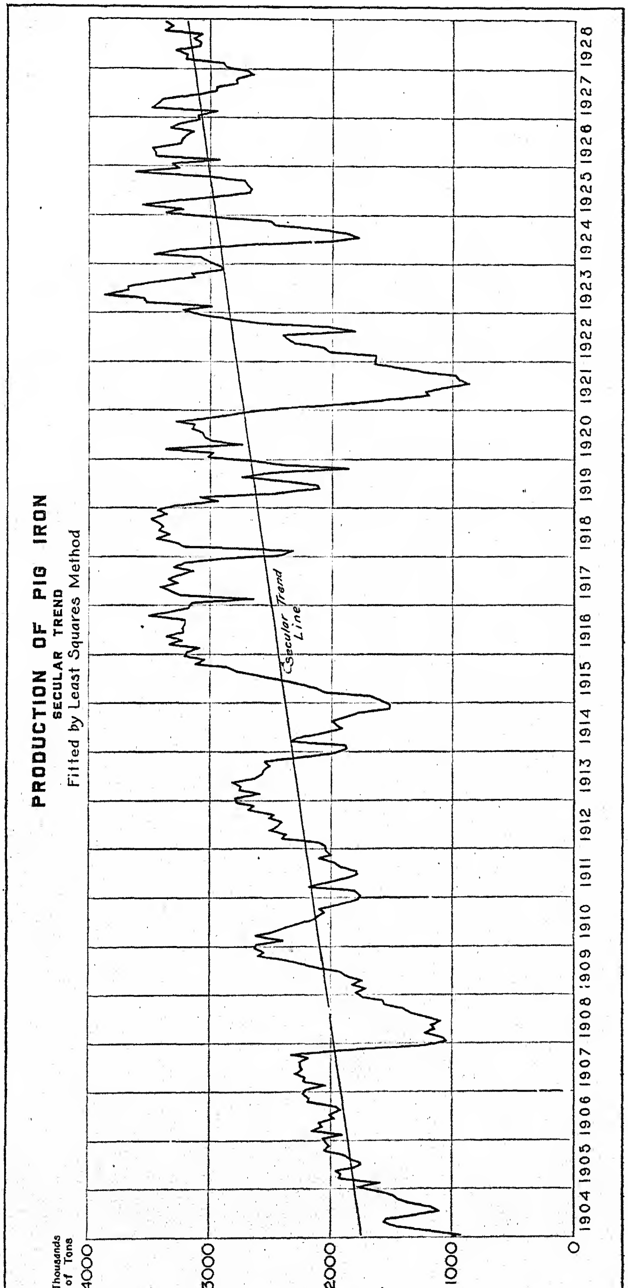

PRODUCTION OF PIG IRON
SECULAR TREND
Fitted by Least Squares Method
Thousands of Tons
4000
3000
2000
1000
0
1904 1905 1906 1907 1908 1909 1910 1911 1912 1913 1914 1915 1916 1917 1918 1919 1920 1921 1922 1923 1924 1925 1926 1927 1928
Secular Trend Line

FIG. 39.

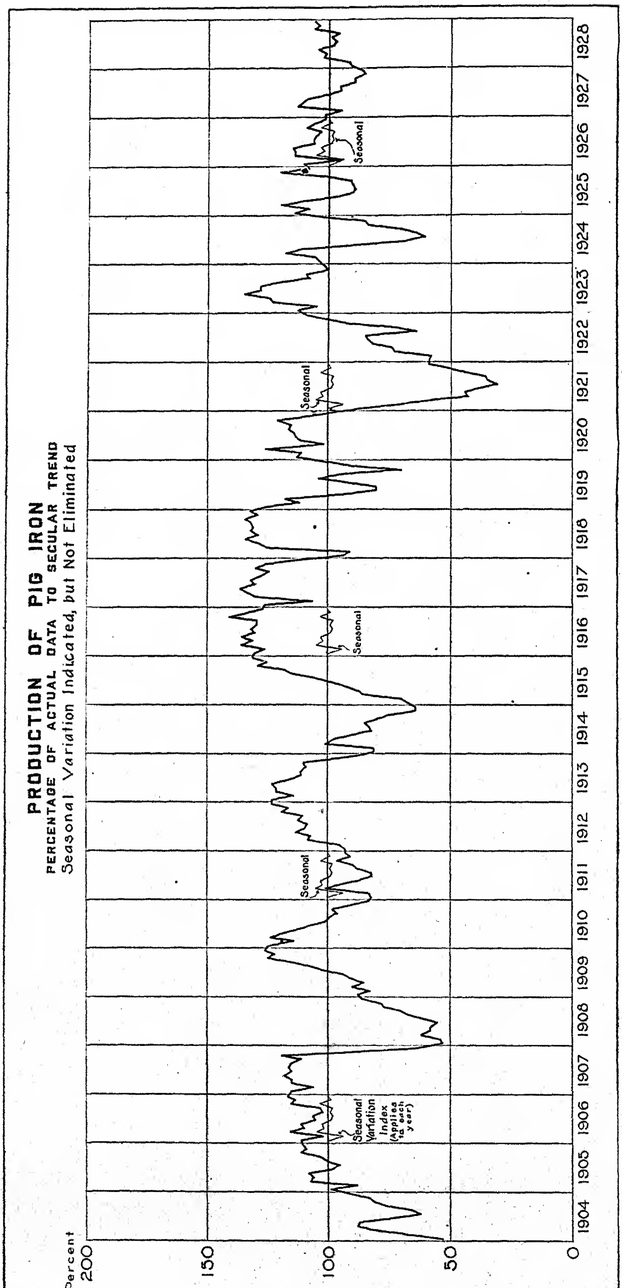

PRODUCTION OF PIG IRON
PERCENTAGE OF ACTUAL DATA TO SECULAR TREND
Seasonal Variation Indicated, but Not Eliminated
Percent
200
150
100
50
0
Seasonal Variation Index (Applies to each year)
Seasonal
Seasonal
Seasonal
Seasonal
1904
1905
1906
1907
1908
1909
1910
1911
1912
1913
1914
1915
1916
1917
1918
1919
1920
1921
1922
1923
1924
1925
1926
1927
1928

FIG. 40.

which the business man tries to isolate quantitative measures for each of the four elements indicated in the previous paragraph. The second is the process of synthesis. It is used to make a forecast based on the assumption that the trend and seasonal, when projected into the future, indicate a normal expectancy. This is in turn modified by the estimated cyclical factor.

Let us assume that we want to analyze a time series and let us select for our series the one which represents the monthly production of pig iron in the United States. Let us view the various steps in the process of analysis before we attempt to calculate specific values. The first step is to remove the trend which represents the march of progress over a long period of time. Figure 39 gives a picture of the actual figures in graphical form for the production of pig iron. Obviously, the trend in this case is upward. A straight line has been selected to represent it. Table 20 gives an illustrative example showing actual values and trend values for the year 1927. The effect of the trend on the series of data is eliminated by expressing the actual value for each month as a percentage of the trend value for that month, as illustrated in Table 20. Obviously, the 100% value represents the line of average tendency, above and below which the percentages are distributed. By using the trend values for the base, the trend line is turned to a horizontal position, and these percentages are plotted above and below it. The result is shown in Fig. 40.

The next step in the analysis is to determine a measure of seasonal variation which reflects the changing mood of the seasons. These measures, or indexes, should give a value for each month in the year which has recurred persistently for that month year after year. For pig iron production, the seasonal variation is relatively small; that is, there is only a slight variation from 100%. The values for the seasonal indexes are given in Table 20 and are plotted in Fig. 41. Just as in the case of the trend line, methods of calculating these indexes will be discussed later. After the indexes have been calculated, the obvious thing to do is to remove their influence from the per cent of trend figures. The need for the correction may be seen by comparing the fluctuations shown in the percentages of actual to trend with the seasonal indexes as drawn in Fig. 41. It is probably better to make this correction by dividing each monthly percentage value, corrected for trend, by the corresponding index for seasonal variation, rather

than to accomplish this step by a process of subtraction. Conditions under which the subtraction process may be used are discussed on page 170.

The plotted line, after the effects of both trend and seasonal variation have been removed, shows fluctuations in percentage form above and below the 100% line. These percentages are called "cycle relatives." The results are shown in Fig. 42. The 100% line which goes through the middle of the chart is called "normal." It represents the values formed by the com-

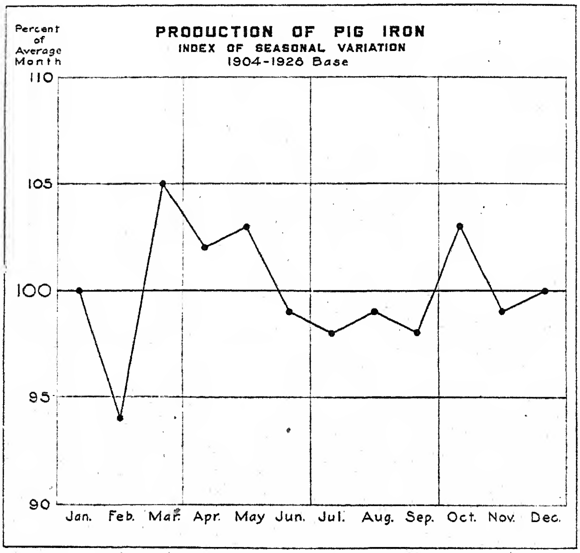


FIG. 41.

bined forces of both trend and seasonal. Care should be taken when using the word "normal," since many statisticians use it in the sense of the trend value only; occasionally, however, it is used with other meanings.

The fluctuations which now remain in the curve represent the increase or decrease of activity in the production of pig iron together with the added effects of accidental fortune. If the curve is examined, times of depression as well as those of marked prosperity in the business history of the industry easily can be discerned. Since there is no known method of separating the cyclical fluctuations from the accidental influence, about the only thing that can be done to lessen the effect of accidental fluctuations

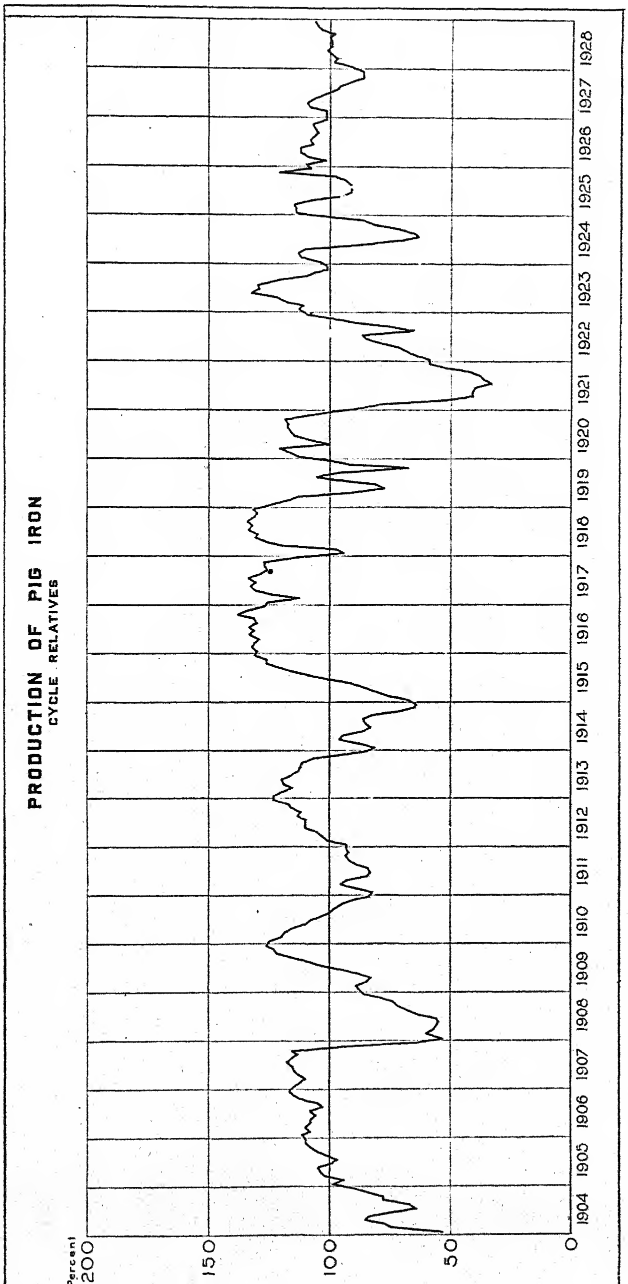

PRODUCTION OF PIG IRON
CYCLE RELATIVES
Percent
200
150
100
50
0
1904
1905
1906
1907
1908
1909
1910
1911
1912
1913
1914
1915
1916
1917
1918
1919
1920
1921
1922
1923
1924
1925
1926
1927
1928

FIG. 42.

is to smooth off the somewhat rough appearance of the chart. This may be done by means of a moving average (see page 170).

At times it may be desirable to compare the chart of cyclical fluctuations in one industry with that of some other business. In order to do this it is necessary to put the two series on a comparable basis. This requires an additional step because the cyclical fluctuations in one business may be characteristically wider than those in another business; yet each may be typical of its own business. Hence, the typical fluctuations which differ for the two series must be made comparable. This can be accomplished by means of the standard deviation of the cycle relatives which represent the typical fluctuation of each series. After the standard deviation has been calculated, the difference between each cycle relative and normal is divided by the standard deviation. The result will be a series of cyclical deviations expressed in terms of the standard deviation for that series. This series may now be compared with another series which has been similarly expressed in terms of its standard deviation because both have a common denominator.

TABLE 20

Pig Iron Production, 1927

Actual Data and Calculated Values for Trend, Seasonal, and Cycle Relatives

(Unit: 1,000 Gross Tons)

Date	Actual Data	Trend Values	Actual as Per Cent of Trend	Seasonal Index 100 = Average Month	Cycle Relatives Per Cent of Actual to Normal
1927					
January	3,104	3,077	101	100	101
February	2,941	3,082	95	94	101
March	3,483	3,087	113	105	108
April	3,422	3,092	111	102	109
May	3,391	3,097	109	103	106
June	3,090	3,101	100	99	101
July	2,951	3,106	95	98	97
August	2,947	3,111	95	99	96
September	2,775	3,116	89	98	91
October	2,784	3,121	89	103	86
November	2,648	3,125	85	99	86
December	2,696	3,130	86	100	86

The table gives values for the year 1927. Figures 40, 41, and 42 are based on a complete series for the period 1904 to 1928.

CHAPTER VI

TREND LINES

"The trend of sales of this power company is a straight line which shows that the sales by 1940 will be more than double the present sales," said W. A. Hastings to Robert Jones of the W. R. Jones Company, industrial engineers.

His friend did not seem to be overimpressed.

"Why, Bob, this can't be far wrong because we have used all of the monthly figures since the company was organized five years ago. The trend was determined by the method of least squares and the calculations have been checked," continued W. A. Hastings.

Jones smiled.

"The chances are excellent," he said, "that by 1940 your forecast will be far out of line unless you are exceptionally lucky. In addition it is likely that the trend for the five years of the company's life which you think is so good, is at best only a fair approximation. You don't seem to realize that you picked out your answer in advance when you selected the straight line to represent the trend, so that all your mathematics did for you was to adjust and adapt your own preconceived ideas to the data."

The facts brought out in this conversation are not at all unusual. Jones knew the difficulty of determining a trend summary for a given problem.

It is altogether too often that the unsatisfactory picture of a trend, which a mathematically fitted line frequently gives, is overlooked. In a determination of any trend, two elements are involved. One is the act of selecting a particular type of line; the other is the process of adjusting the line selected to the given data. The selection of a curve, when a mathematical method is used, is equivalent to the selection of a particular equation which represents that curve. Such equations, which are convenient to use, are very limited in number so that the choice of trend in turn is strictly limited. The second process, the adjustment of the selected equation or line, is made in strict accordance

with the mathematical assumptions which are involved in the particular method selected for the adjustment of the equation to the data. The trend line derived by this arbitrary process has one advantage: the rigidity of the mathematical law provides an easy method for making a projection. Such a projection may or may not present a satisfactory expression of the future trend.

The two fundamental elements indicated have been discussed with relation to the mathematical methods of determining a trend line. Before the technique of these methods is outlined, it is desirable that we should consider these same two elements in relation to the graphical method of determining trend.

In the case of the graphical method, the two elements indicated above appear in quite a different guise. First, the selection of a curve is not restricted. This is because the curve is drawn by eye or with the partial guide of a curved ruler. The mathematical equations which will correspond to the vast majority of such curves would be very complicated. Yet by a graphical method they may be simply drawn. Selection, therefore, is very flexible. In the next place, selection of a curve does not precede the process of fitting but is concurrent with it. The line thus fitted represents the result of a compromise between the statistician's conception of a good fit and his judgment as to what the general nature of the trend should be for various periods of the data. Although a line drawn graphically may be extended beyond the limits of the data, the projection will be determined partly by the general direction of the fitted line and partly by judgment. The projection is not determined here by the standards of a mathematical law. A personal bias is allowed to enter which makes the graphical method in this respect less satisfactory.

From another point of view, the graphical method is an advantage to the statistician because it allows him a greater freedom for the exercise of judgment. It is assumed that this is an advantage because his judgment will be based not only on a thorough understanding of the particular set of data involved, but also upon an understanding of the mathematical methods which would have to be used in case a mathematical procedure had been adopted. The result of these general observations is that the selection and adjustment of a trend line to a particular set of data involve, to a considerable degree, skill and judgment. In philosophy, the idea of trend line is simple to understand; the

mathematical laws which underlie it, however, are by no means simple.

Since no definite rules for the selection of a particular kind of trend are known, four different types of trend lines will provide sufficient variety to cover the more important possibilities. These four types of trend are the straight line, the compound interest curve, the parabola, and the Gompertz curve. It will be found that each of these trend lines has certain properties that commend it, as well as certain properties that make it unsatisfactory for use in particular cases. Before the method of fitting each of these curves to a series of data is discussed, it will be wise to acquire an understanding of certain characteristics of each.

The straight line needs no introduction. Applied to data it represents a straight inflexible path. If the trend of the data is really curved, then the straight line will represent a cord or a secant cutting across the curve. For short series its simplicity gives it a natural advantage. For long series its inflexibility often makes it undesirable. When forecasting, we are forced to project the trend beyond the latest data available, so that the straight line becomes increasingly questionable with the increasing length of the period of the forecast. Thus, for example, if we plot our weight by years from the age of 5 up to the age of 10 and draw a straight-line trend through the data plotted, the projected straight line probably would not appeal to us as a satisfactory forecast of our weight when we reach the age of 60. To make the illustration definite, if our weight increased from 40 pounds at age 5 to 100 pounds at age 10, then by a straight-line trend we should weigh 700 pounds at age 60.

The compound interest trend may not be as well known. This curve gets its name from banking mathematics. If a sum of money is put in a savings account and allowed to accumulate at compound interest, the amount at the end of each compounding period can be calculated. If the calculated amounts corresponding to each period are plotted, the plotted points will be arranged in the form of a curve, as shown in Fig. 43. As has been pointed out on page 29, this compound interest curve when plotted on a semilogarithmic chart becomes a straight line. The compound interest trend is exceptionally useful for interpolation (determination of trend values within the limits of the data) because in many businesses such data as the sales over a short period often exhibit a compound interest rate of growth. On the other hand, its use

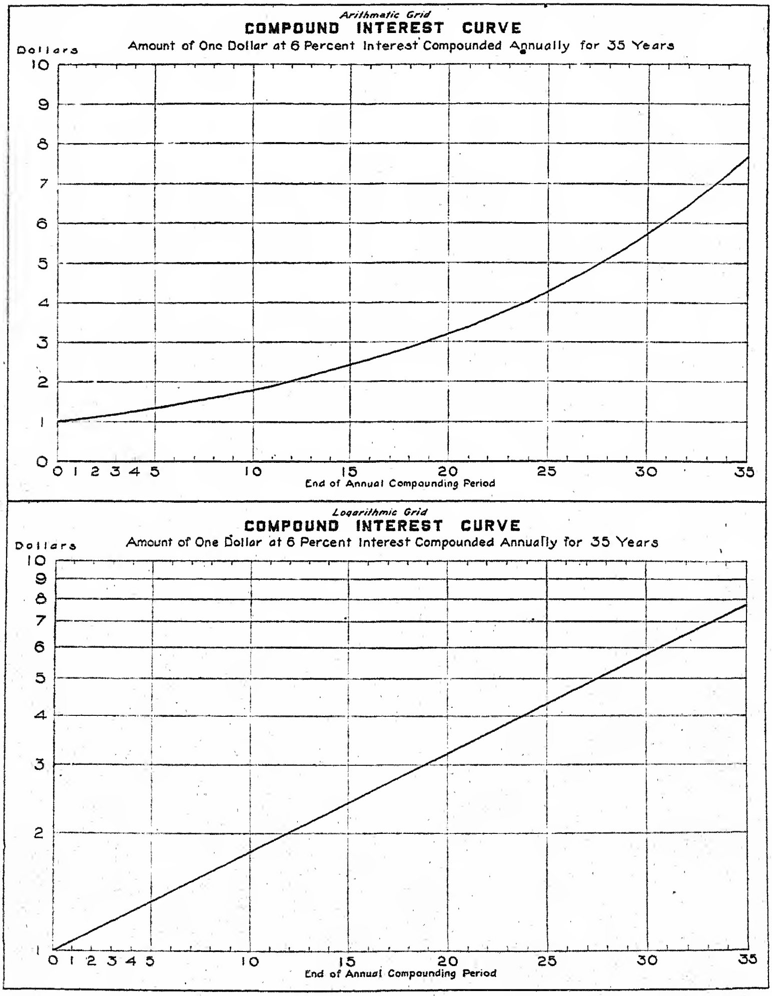

Arithmetic Grid
COMPOUND INTEREST CURVE
Amount of One Dollar at 6 Percent Interest Compounded Annually for 35 Years
Dollars
10
9
8
7
6
5
4
3
2
1
0
0 1 2 3 4 5 10 15 20 25 30 35
End of Annual Compounding Period
Logarithmic Grid
COMPOUND INTEREST CURVE
Amount of One Dollar at 6 Percent Interest Compounded Annually for 35 Years
Dollars
10
9
8
7
6
5
4
3
2
1
0 1 2 3 4 5 10 15 20 25 30 35
End of Annual Compounding Period

FIG. 43.

for extrapolation (determination of trend values for future years beyond the limits of the data) is limited. The available forecasted values reach absurdly high values even sooner than the forecasted values for a straight line since the compound interest curve rises very rapidly.

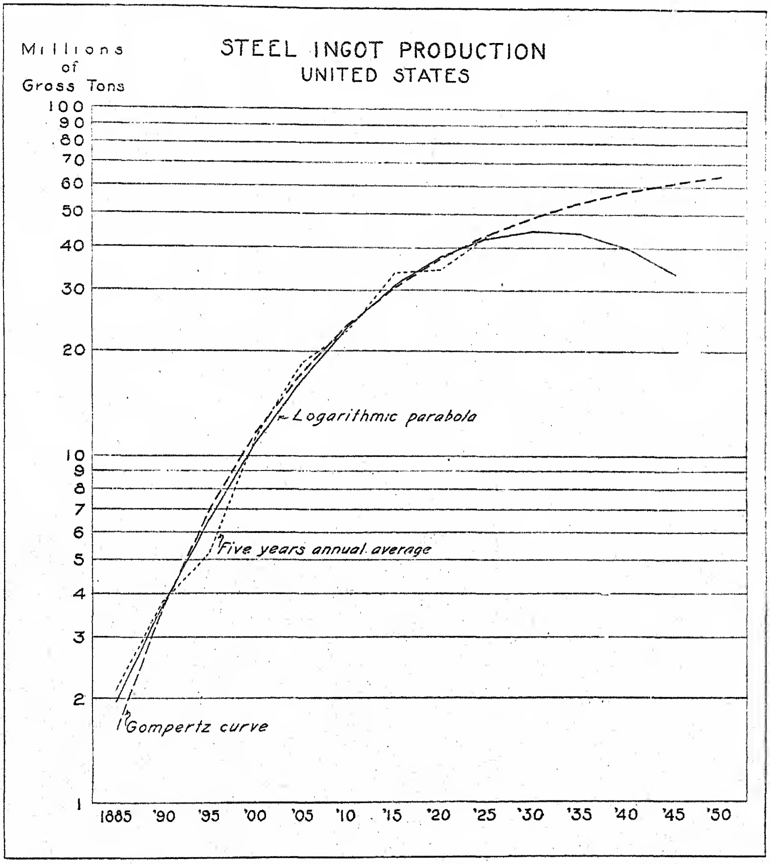


FIG. 44.

The next more complicated trend is represented by a parabolic arc. The shape of this curve is familiar to everyone since it is the curve of an automobile headlight reflector. An example is shown in Fig. 44. This curve is plotted on a semilogarithmic grid. The same curve might have been plotted on an arithmetic grid. It will be noticed that the arc of the curve is concave down-

ward; that is, the axis of the parabola is parallel with the vertical axis of the grid. If the arc had been projected further, the curve finally would have approached the zero value. Extended in this way the curve would have been symmetrical on both sides of its vertical axis. On the other hand, the constants of the equation may have signs reversed so that the curve has its axis pointing upward. It then would be located on a grid much like the outline of a bowl. From these facts and an inspection of the figure, it may be concluded that the curve is satisfactory for some curvilinear trends within the range of the data, but that it is quite unsatisfactory for long-term forecasting. The value of the curve for many practical applications consequently is limited. However, in many cases it does give a good approximation of the trend.

In order to provide a line which is satisfactory for long-term forecasting, the so-called Gompertz curve has been devised. This is an elongated S curve which starts close to the horizontal base line, rises slowly at first, then faster, then more slowly, and finally levels off against an upper "ceiling," to use an aviation term. This curve is supposed to represent the history of an industry which grows slowly at first, then rather rapidly, and finally begins to grow more slowly as demand for its products approaches a condition of saturation in the market. At first sight it may seem that the Gompertz curve is exactly the thing which is desired for a trend line. The difficulty is, of course, that the data too often fail to reflect the condition just described.

CUMULATIVE AND NONCUMULATIVE DATA

Before we consider the methods used to fit the curves discussed above, we must understand the difference between cumulative and noncumulative data which is often overlooked because it appears in the form of a tacit or implicit assumption. Since the data hold up no red flag to warn the analyst, the recognition must be made by the individual. Series which represent prices or inventory are noncumulative because they picture conditions on a given day, and because the price or inventory for each day is to be considered independently of that of a previous day. Series representing volume of sales, on the other hand, are usually in the cumulative form because they represent the accumulation of sales day after day for a period of a month or, in some cases, for a period of a year.

Noncumulative data such as inventory are considered characteristic of a certain day; consequently, when noncumulative series are plotted, they should be plotted at a point corresponding to that day. For example, if inventories are estimated as of the end of the month, they should be plotted on the line dividing that month from the succeeding month. On the other hand, average monthly inventory should be plotted in the middle of each year when each average represents a year as a whole.

Cumulative data, such as monthly sales, represent the work which has gone on throughout a whole month. The figures are characteristic of the month as a whole. It is better practice, therefore, to plot these figures in the center of the spaces reserved for particular months. This has been described above on page 21.

In connection with the calculation of trend lines, the difference in the two types of data is rather confusing unless the cumulative data are reduced to a noncumulative form. This can be done by a simple division. If it is stated, for example, that one year's sales amount to $60,000, the implication is that these cumulative data are in cumulative form. The same year's sales may be expressed in noncumulative form as one-twelfth of the original amount, or $5,000. Usually this is called the "monthly average" sales for the year. Thus, the distinction between the cumulative and the noncumulative form is the same as that between a total and an average. The noncumulative type of data, such as inventories, obviously cannot be presented in the form of totals, but the cumulative series, such as sales, can always be reduced to their averages. For that reason, in the following examples both types of data will be put on a common basis by expressing the cumulative sales in noncumulative form. The method of calculation from the data in cumulative form will be described at the end of this section.

CALCULATION OF STRAIGHT-LINE TREND

Consideration will now be given to a method of fitting a straight-line trend to a series of data. We shall take as an example the noncumulative data for inventory for the Morgan Department Store case, as shown in Table 22. Adjustment of the straight line trend involves a mathematical process which makes use of the observed values as shown in the table of the data. The method of using these values includes a statement of an equation

of a straight line. We shall take as our equation $Y = a + bX$. An equation of this form which has the first power of Y and the first power of X is a straight line when plotted on an arithmetic grid. The letters a and b represent the values that determine the position of the line, as we shall see. When the values of a and b are known, we select some value for X, and calculate from the equation the corresponding value for Y. This gives us the quantity which is the coordinate on the Y-axis of the point corresponding to the given value on the X-axis. A series of points determined in this way always lies along a straight line. The reverse is also true. Any straight line which may be drawn on an arithmetic grid can be represented by an equation of the form given.

Let us now see what the letters a and b represent. When $X = O$, the equation tells us that $Y = a$. The value a is technically known as the Y-intercept because it is the height of the point on the Y-axis where the line cuts it. We can think of this as a starting point on the Y-axis. Let us now imagine that X is successively equal to 1, 2, and 3. Then we add to the value a, respectively, b units, $2b$ units, and $3b$ units. In other words, the addition of one unit of X increases Y by b units. The value of b is technically known as the increment.

Our problem in adjusting the line to the data, then, is equivalent to the problem of finding the values of a and b. The method used here to determine these values is the so-called method of least squares.

The *method of least squares* is derived from the theory of a normal frequency distribution. Among other things, the method assumes that the line is best adjusted to a series of observations when the sum of the squares of the deviations of the observed points from the line is less than that for any other line which may be drawn. The detailed development of the theory is omitted here, since it involves considerable mathematics. Application of the formulae which are derived by the mathematics, however, is comparatively simple. Briefly, the theory shows that the constants a and b may be determined by solving two equations which are known as "normal equations." These are:

$$\Sigma Y = Na + b\Sigma X$$
$$\Sigma XY = a\Sigma X + b\Sigma X^2$$

where N represents the number of observations and the Greek letter sigma (Σ) represents the sum of the quantities to be taken as indicated.

It is obvious that in time series, which present annual data, the values of X can be the number of the year such as 1926 or 1930. This, however, makes altogether too much number work since the numbers representing years have four digits. A first simplification, obviously, is to make the zero year correspond to the first observed value. This will be entirely satisfactory for use with these equations. If, however, no term is missing in the series, the mid-date of the series may be selected to represent the zero year, with the years before the zero year shown as negative years, and those after as positive years, so that the sum of the X's will be equal to zero. Consequently, two terms in the above equations will drop out. The normal equations in the simplified form then become

$$\Sigma Y = Na$$
$$\Sigma XY = b\Sigma X^2$$

so that the values of the constants under these conditions are

$$a = \frac{\Sigma Y}{N}$$
$$b = \frac{\Sigma XY}{\Sigma X^2}$$

Let us now return to the noncumulative data representing inventory for the Morgan Department Store as shown in Table 22. To simplify the calculations let us use an odd number of years. The values of Y correspond to the annual inventory figures. The values of X indicate the years. The middle year is numbered O, with negative numbers for the years earlier than the middle year, and positive numbers for the years later. The products for XY and X^2 then are calculated and the sums determined. The annual increment of trend is given by $\Sigma XY/\Sigma X^2$. In this example, the annual trend increment, b, has the value of 0.177.

The next things to determine are the trend values for different months. The a value is obtained by $\Sigma Y/N$ which gives 3.667. Since we are using the middle year as origin, this is the average of the original Y values. This figure represents the annual (noncumulative) trend value of 1924, the middle year, and also the monthly value of the middle month of the same middle year. This is not a calendar month, for it includes the latter half of June and the first half of July. In order to adjust this figure,

3.667, to the calendar month of July it is necessary to increase it by one-half of the monthly increment (see Table 23, page 173).

The monthly trend increment is obtained by dividing the annual increment by 12 because there are 12 monthly increases from the trend value in one year to the trend value in the next year. The value of this monthly increment is 0.177/12 or +0.015. The positive sign means that the slope of the trend line is upward. Hence we add one-half of the monthly increment, or 0.008, to the mid-monthly trend value of 3.667 to obtain the inventory value for July, which is equal to 3.67. The trend value for any other month can be obtained by adding to or subtracting from this value of 3.67 the monthly increment multiplied by the corresponding number of months. For example, Table 23 shows that the inventory value for February, 1922 is 3.67 minus 29 times the monthly increment, 0.015. This equals 3.23.

The steps in the calculation can be summarized as follows:

SUMMARY

(Odd number of years, the middle year taken as origin, noncumulative data)

1. Calculate the average value of Y, which gives the mid-monthly trend value of the middle year.

2. Determine the annual increment and divide it by 12 to obtain the monthly increment.

3. In order to obtain the trend value for the month of July, correct the average Y-value by adding one-half the monthly increment.

4. Determine the trend values for other months by adding to or subtracting from the value determined in step 3 the monthly increment multiplied by the corresponding number of months.

When we use the Morgan Department Store data to calculate trend for sales (see Table 25) we are dealing with cumulative data.[1] The first step here is to express the sales in noncumulative form. This is done by dividing each annual figure by 12. Then the data will be on a common basis with inventory, that is, on the basis of monthly averages. The rest of the calculations are exactly like those explained above (see Table 27).

[1] See "Special Case" on page 162.

When the data have an even number of years, the middle or zero point obviously comes half-way between the two middle years. This would make the deviations from the zero point to the midpoints of the two middle years only one-half year. The two years next to these middle years would have to be designated as one and one-half years, and so on. Since fractional values are awkward, for purposes of calculation each X-value is multiplied by 2. Thus, we have instead of X the value of $2X$. The corrections in the example as shown in Table 26 are obvious. After the values for the average Y and the annual increments have been calculated, the rest of the process is identical with that already described.

CALCULATION OF COMPOUND INTEREST TREND

We have seen that a compound interest curve on a semilogarithmic grid is represented by a straight line. A straight line may be represented by the equation $Y = a + bX$. Therefore, a compound interest curve may also be represented by an equation of the same type if the Y-values are plotted on a semilogarithmic grid or expressed in logarithmic form. The equation of the curve may be given as $\log Y = A + BX$, where A is the logarithm of the intercept, a, and B is the logarithm of the ratio of increase, b.[1] In order to calculate the trend values it is necessary first to find, in the table of logarithms, the logarithms of the original Y-values, and use them instead of the original figures. The values of A and B will be determined by the method of least squares already explained above.

From the values, A and B, the trend values on the straight line may be obtained as before. These will represent, of course, the logarithms of the annual trend values. The corresponding natural numbers may be found from the table of logarithms.

CALCULATION OF A PARABOLIC TREND

A parabola is represented by an equation of a type similar to that for the straight line. A term with an X^2 is added so that the equation becomes $Y = a + bX + cX^2$. The calculation of an equation for a parabola from a series of observations may be done by the method of least squares. The normal equations used

[1] The equation of a compound interest curve in natural numbers is $Y = ab^X$, which in logarithmic form becomes: $\log Y = \log a + X \log b$

in the case of the straight line can be extended to cover this case also. These normal equations are

$$\Sigma Y = Na + b\Sigma X + c\Sigma X^2$$
$$\Sigma XY = a\Sigma X + b\Sigma X^2 + c\Sigma X^3$$
$$\Sigma X^2Y = a\Sigma X^2 + b\Sigma X^3 + c\Sigma X^4$$

If the zero value for X is located at the midpoint so that the sum of the X and the sum of the X^3 is equal to zero, these equations reduce to the simpler form

$$\Sigma Y = Na + c\Sigma X^2$$
$$\Sigma XY = b\Sigma X^2$$
$$\Sigma X^2Y = a\Sigma X^2 + c\Sigma X^4$$

These normal equations determine a parabolic trend for a series of data in natural numbers. The calculation of a parabolic trend line may be made from the logarithms of the Y-values instead of the natural values of the Y. The equation becomes $\log Y = a + bX + cX^2$. Logarithms of the trend values can be calculated by the method explained above if "log Y" is substituted for "Y" in the normal equations. The equation for the values of X gives the logarithms of the trend values, that is, the "log Y." The corresponding Y-values in natural numbers can be found by looking up the numbers in a table of logarithms.

For many series of data a parabolic arc on a semilogarithmic grid seems to be exceptionally desirable. In practice there are many basic series of data of broad economic significance covering a long period of years for which a parabolic arc on a semilogarithmic grid seems to present an excellent picture of what we believe the trend should be.[1] Because the parabolic arc in such cases is usually quite flat, it may be used as a curve of a reasonably short projection into the future. The curve does not begin to turn downward from the highest point until we have given a considerable number of time intervals beyond the data. The calculation of such a trend line is given in Table 35 and the curve is shown in Fig. 44.

CALCULATION OF A GOMPERTZ CURVE

The Gompertz curve is handled most easily from the equation of the curve in its logarithmic form. This is $\log Y = \log a + c^X \log b$.

[1] See, for example, Snyder, Carl, *Business Cycles and Business Measurements*, Chapter II.

It will be observed that three unknown constants, a, b, and c, have to be determined. Before the calculation is begun, a comment or two in regard to these constants may be of assistance. The quantity log a represents the upper limit beyond which the curve does not go. It gradually approaches this limit which may be thought of as a "ceiling." The second term of this equation, $c^x \log b$, represents the amount deducted from this ceiling in the calculation of individual trend values. Consequently, since the value is a deduction, it must be negative. For curves showing an increasing growth the log b is negative. The value of the constant c controls the amounts that should be cut off from the ceiling at various points. Since c is raised to x power, and since diminishing amounts are cut off as x increases, c must be less than unity. In a great many examples it will be found that this value is in the neighborhood of 0.8 or 0.9.

In fitting the curve, an even multiple of three terms must be available with no terms omitted; that is, the data must be divided into three equal blocks of terms. In the example given on page 187, it happens that there are three terms in each block, but this is not necessary as there may be four or ten or any other number. The logarithms of the observed values are found from the table. The sums of these logarithms for each block then are determined. The sum of the logs for the first block is called S_1 and those for the second and third, S_2 and S_3, respectively. If the number of terms in each block is equal to n, then the equations for determining a, b, and c are as follows:

$$c^n = \frac{S_3 - S_2}{S_2 - S_1}$$

$$\log b = (S_2 - S_1)\,\frac{c - 1}{(c^n - 1)^2}$$

$$\log a = \frac{1}{n}\left(S_1 - \frac{c^n - 1}{c - 1}\log b\right)$$

It will be noticed that the value of c must be found from the logarithm, but that the logarithm of b is used in its logarithmic form as is also the logarithm of a. Trend values are given for the log Y as in Table 36 and the actual values of Y are found from a table of logarithms (see Fig. 44).

SPECIAL CASE[1]

Calculation of Monthly Trend Values on a Straight Line from Cumulative Data in Cumulative Form

When the sales figures are given in cumulative form, that is, as totals rather than as averages, there are two points of difference in the procedure as compared with the steps described above. The first difference is that it is necessary to divide the average Y-value by 12 in order to obtain the trend value of the mid-month of the middle year. Then this mid-monthly value should be adjusted as before to the calendar month of July by adding one-half of the monthly increment. The second difference is that the annual increment in total sales should be divided by 144 instead of by 12 to determine the monthly increment.

In order to see the reason for the second difference in procedure, it is necessary to understand the distinction between the annual increment in total sales and the annual increment in average monthly sales. Note that cumulative data, such as sales, may be presented in either of two forms, cumulative or noncumulative. In its cumulative form an annual figure represents a *total* of monthly values. In a noncumulative form, however, it is an *average* of monthly values. The *totals* would be 12 times larger in volume than the corresponding *averages* of the same series. If the series represents a straight-line trend, the constant difference between any *total* figure and the preceding one is known as the annual increment in *cumulative* form. Similarly, the constant difference between any *average* and the preceding one may be called the annual increment in *noncumulative* form. The former type of increment is 12 times as large as the latter because it represents the constant increase in the total for a year while the latter represents the corresponding increase in the average for the same year. The latter increment is equivalent to the increase in volume from any month of any year to the same month of the next year, which means that it contains 12 monthly increments. In other words, the annual increment in total sales is 12 times as large as the annual increment in average sales, and the latter in turn consists of 12 monthly increments. Therefore, when the cumulative data are used in cumulative form the annual increment is equivalent to 144 monthly increments.

[1] This special case was referred to on page 158.

The following numerical example illustrates the three different types of increment. The first two are annual and the third one is monthly. The series represent sales by months for two years, beginning with 14 units in January of the first year, and increasing each month by 5 units.

If the following series were not sales, but prices, inventories, population, or some other type of series which would not allow a cumulation of monthly values, the annual values would have to be represented as averages. Then there would be only two types of increment since the first of the three would have no meaning.

TABLE 21

Month	Sales for One Year	Sales for Next Year	Difference
January	14	14 + 60	60 = 12 × 5
February	14 + 5	14 + 65	60 = 12 × 5
March	14 + 10	14 + 70	60 = 12 × 5
April	14 + 15	14 + 75	60 = 12 × 5
May	14 + 20	14 + 80	60 = 12 × 5
June	14 + 25	14 + 85	60 = 12 × 5
July	14 + 30	14 + 90	60 = 12 × 5
August	14 + 35	14 + 95	60 = 12 × 5
September	14 + 40	14 + 100	60 = 12 × 5
October	14 + 45	14 + 105	60 = 12 × 5
November	14 + 50	14 + 110	60 = 12 × 5
December	14 + 55	14 + 115	60 = 12 × 5
Totals	168 + 330 = 498	168 + 1,050 = 1,218	720 = 12 × 12 × 5 = 144 × 5
Averages	41.5	101.5	60

Summary of table:

Sales	One Year	Next Year
Total Annual Sales (Cumulative Form)	498.0	1,218.0
Average Monthly Sales (Noncumulative Form)	41.5	101.5

Three Types of the Increment:

1. Annual increment in the total sales 720
2. Annual increment in the average sales 60
3. Monthly increment in sales 5

Table 28 illustrates the method of calculation of monthly trend values from the Morgan Department Store sales without a reduction of the series to a noncumulative basis.

CHAPTER VII

INDEXES OF SEASONAL VARIATION

Through the computation of indexes of seasonal variation we attempt to obtain tangible measures of the effect of the changing seasons of the year. Less frequently we intend to measure periodic fluctuations within a month, a week, or a day. The most commonly used index reflects the monthly variations which are assumed to be the same from year to year. These indexes usually are expressed in the form of percentages of the imaginary average month. The reader should be constantly on his guard in the use and the interpretation of indexes of seasonal variation. Because of the variety of types of fluctuations in any economic series, it is difficult to segregate any one of the types, and consequently no known method of calculating the seasonal index gives absolutely accurate results. In spite of this fact, however, people often refer to seasonal indexes in such terms as to indicate that the indexes carry the weight of exactness and of authority.

Two different averages are used in the calculation of seasonal indexes. In order to understand the application as well as the construction of the indexes, it is necessary to keep these two averages clearly in mind. The first type of average determines a typical figure for each month over the whole period of years. This gives a set of 12 typical figures. The second type is a "cross-section" average of these 12 figures; it determines the base to which these 12 figures should be related in order to be converted into percentage form. This second step is desirable because the business man likes to think of each month in terms of the average month of the year. Such a base marks a level against which he can check good or bad months. Consequently, the second "average" used in calculating seasonal indexes is the arithmetic mean of the typical figures for 12 months. By dividing each typical figure by this mean we express each index in terms of the average month.

In order to illustrate the general principles of calculation of the indexes for a simple case, let us consider the figures representing temperature for some given city in the United States.

If the data are available for a period of 10 or 12 years, a typical figure for January can be obtained by averaging all the Januaries to get the typical figure for January, a second typical figure can be obtained for February, and so on. It is to be noted that these typical figures are not the figures for any one year but are simply representative of each particular month for all years. Moreover, it will be recalled that any kind of an average represents the central tendency of a frequency distribution. Here, the arithmetic average is used to represent the distribution of the figures by years for each month. The set of 12 typical figures obtained by this process is a perfectly satisfactory index of seasonal variation. It will be recalled that we have said that a seasonal index implied two averages. In this simple case, however, the second average is not necessary. Trend is lacking and the figures are in original units so that the normal for each month is the index for that month.

In dealing with the problems which usually occur in business, the statistician finds a much more complicated situation. These complications occur because of the effects of trend and cycle which were not present in the case just cited. The effects of trend and of cyclical fluctuations possibly could be reduced greatly by averaging, provided enough terms for each month were available. It is unfortunate that, even though a sufficient number of years were available, usually it would not be satisfactory to use them because over a period of more than about 7 years the seasonal variation for almost any business series changes sufficiently to bring in added changes. The variation in monthly sales in certain departments of department stores, for example, has shifted radically during the past few years. Since the statistician wishes to avoid this trouble, he limits himself to 7 to 9 years. Consequently, typical values have to be calculated from a frequency distribution containing only 7 to 9 terms. These terms often have such scattered values that there is very little evidence of concentration. This means that any average figure which may be chosen as typical may not be truly representative.

Three methods of calculating indexes of seasonal variation will be discussed. Whether one or another of the methods should be used depends entirely upon the character of the fluctuations present in the data. The difficulties which have been enumerated in securing a typical figure should always be borne carefully in mind when using any particular method.

The three methods which will be described are the moving average method, the per cent of trend method, and the link relative method.

In applying the *moving average method* the steps are:

1. Find a total of the figures for the first 12 months of the series; then a total for the 12 months which omits the figure for the first month and adds the figure for the thirteenth month, and so on.

2. Divide moving totals by 12 to obtain a moving average. Usually each figure in the moving average is centered at the mid-point of each group of 12 months; that is, on the first day of the seventh month.

3. Divide each of the original figures by the corresponding value for the moving average and thus express them in per cent of the moving average.

4. Find a typical figure for January, one for February, and in turn one for each of the other months. This is done by arranging the January relatives, the February relatives, and so on, in 12 vertical columns. Each column may be regarded as a frequency distribution. Choose as the typical figure either the average of the middle-sized three or four, or the median.

5. In order to conform to a more convenient standard, divide each of the 12 typical figures by their arithmetic average and multiply the results by 100. This will give a seasonal index with each month expressed as a percentage of the average month. The 12 values of the index should total 1,200%.

Theoretically, the moving average used above includes the elements of trend, cycle, and accidental, but not of seasonal influences. Therefore, ratios of the original item to the corresponding moving average are assumed to reflect seasonal influences only. A simple example will explain why the moving average is assumed to contain no seasonal variation. Take a set of numbers such as 9, 6, 5, 3, 7, 9, 6, 5, 3, 7. It is obvious that the numbers repeat themselves after every fifth number. The sum of the first 5 is equal to 30 so that the average of the first 5 is 6. If now we drop the first number, 9, and add the sixth number which is also 9, the sum obviously will remain the same so that the average is still 6. Thus, where a series has a periodicity represented by 5 figures, a moving average of 5 figures will lack that periodicity entirely. Similarly, a 12 months' moving average contains no trace of a 12 months' seasonal variation. The theory used is

nice, but practice does not always follow. Cyclical and accidental fluctuations often conceal the pattern of the seasonal index by showing a temporary regularity in the nonseasonal swings. The result is that the 12 months' moving average either fails to eliminate all of the seasonal effect or slightly overdoes the matter.

Of course, this defect can be removed by using a sufficient number of years, but, as pointed out above, the character of the seasonal variation might change, necessitating indexes of changing seasonal variation.

The *per cent of trend method* as far as mechanics are concerned is similar to the moving average method, but the philosophy is somewhat different. Let us consider first the steps that are involved. These are:

1. Divide the value for each month by that of the corresponding trend value. Express the result in percentage form.

2. Arrange the January figures, the February figures, and so on, in 12 vertical columns each of which may be regarded as a frequency distribution.

3. Select from each column a typical or "average" figure. The usual practice is to take the median or a "modified median," that is, the arithmetic average of two or three of the middle-sized figures. This will give a set of 12 typical values.

4. Divide the 12 figures by their arithmetic average and multiply the results by 100.

This method of dividing the data by the trend removes only the element of trend from the series of data, while division by the moving average theoretically removes the cyclical and accidental influences as well as the trend. After the trend has been removed, we hope in the process of averaging to eliminate the effects of cyclical and accidental changes in the data so that we can determine a figure for each month of the year which is typical of the seasonal variation. It is at once obvious from our discussion that this is not always accomplished.

A third method of calculating the indexes is the so-called *link relative method.* This depends upon the ratio of the number corresponding to each month to the number corresponding to the preceding month. Although no trend is indicated, the method to some extent is predicated on a compound interest trend which is commonly found in economic data. The objections to the method are similar to those that have been named in the preceding paragraphs; that is, a short series of cyclical fluctuations acci-

dentally arranged in a proper sequence may influence unduly the link relatives. The steps in the link relative method are as follows:

1. Divide the value for each month by that of the preceding month. Express the result in percentage form.

2. Collect all of the ratios or relatives in 12 vertical columns. Obviously, if the work has been planned effectively, this step is accomplished as soon as the results of step 1 are written down.

3. Select from each column of figures a typical or "average" figure. The median or the average of two or three of the middle-sized figures may be selected as most typical. This will give a single figure for each month, which is the first average referred to above. Theoretically, these median link relatives for each month give a set of seasonal indexes. For practical purposes, however, their use is open to the objection that the base is continually shifting because the value for each month is expressed in terms of that for the preceding month.

4. Change the shifting base to a fixed base by the process of chaining. This is done arbitrarily by setting January equal to 100. The February link relative is unchanged in value. Multiplying the February link relative by the March link relative gives the new March value, and so on. The effect of this process of chaining is to relate each month to January, which is the constant or fixed base.

5. Correct these chain relatives for the error which is assumed to be distributed in a compound interest fashion over each of the 12 months. The step is made necessary because the December chain relative multiplied by the January link relative should equal the January value which is 100. The result seldom is equal to 100, which would be the case if there were no discrepancies or disturbing elements present in the series. If we are willing to accept the assumption that the error has been accumulated according to a compound interest law or, what is the same thing, if we are willing to say that the error should be distributed among the months in a cumulative percentage manner, then the logical way of making the correction involves the use of logarithms to determine the twelfth root of the total error. This method, however, usually is not necessary. Because the error seldom exceeds 25% of the January base, the difference between the assumptions that the error must be accumulated by a compound interest curve or by a straight line is negligible. Consequently,

the correction for the discrepancy may be made by dividing the total discrepancy discovered at the end of the chaining process by 12. One-twelfth of the discrepancy is then deducted from the chain relative for February, two-twelfths from the chain relative for March, and so on. In case one or more of the seasonal indexes differ materially from 100%, or the cyclical swings are wide, this error should be removed by division rather than by subtraction. Thus, if the chain relatives showed a total error of +24%, you would either subtract 2% from February, 4% from March, and so on, or you would divide February by 1.02, March by 1.04, etc. Of course, if the total error were −24% you would add 2%, 4%, etc., or divide by 0.98, 0.96, etc., according to whether the subtraction or division method of making the correction was used. In general, division is preferable. Of the three methods of correction described above, the second method, which is the process of correction by prorating the amount of the error arithmetically and dividing through, is illustrated in Table 29.

6. Add together the numbers obtained at the end of step 5, divide them by 12, and divide each monthly number by the result. The results of this division should be multiplied also by 100 in order to express the indexes in percentage form. Step 6 is introduced for the practical reason that it gives us indexes related to the average month and not to the first month of the year. In other words, we desire to have the total fluctuations of the months which happened to be above the average (100) equal to those which happened to be below the average. This is the second way of averaging on which seasonal indexes are based.

In order to illustrate calculations of indexes of seasonal variation by the three methods discussed above, Tables 29, 31, and 32 of the Morgan Department Store case have been included.

CHAPTER VIII

DETERMINATION OF THE CYCLE RELATIVES

After the trend and the seasonal have been calculated, the data are corrected for trend by expressing each of the original figures as a percentage of the corresponding trend value as shown in Tables 26 and 30 in the Morgan Department Store case. Next, the seasonal is removed by dividing these corrected figures by the indexes for seasonal variation. In case the indexes for seasonal variation do not vary widely from their average 100% value, and in case the cyclical fluctuations are not large, the effect of the seasonal may be eliminated by a simple process of subtraction rather than by the more complicated process of division. The latter was the method employed in the Morgan Department Store case. The resulting figures give the cyclical fluctuations expressed in per cent of normal as shown in Table 30. These represent the fluctuation of the business cycle as well as the extra or unusual influences that arise from time to time. The fluctuations resulting from these unusual influences may be smoothed out by taking a three or five months' moving average of the cycle relatives, as was done in the Morgan Department Store case.

The cyclical fluctuations may be shown as deviations from normal by subtracting 100 from each relative. In a time series the cyclical fluctuations about the normal can be arranged in the form of a frequency distribution. By expressing the deviations in terms of the standard deviation, the amplitude of the fluctuations may be made comparable with those in another series, provided the other series also is expressed in terms of its standard deviation.

MORGAN DEPARTMENT STORE

The manager of the Morgan Department Store wanted to know whether his store was doing as well as it should in relation to the current business conditions in the city in which it was situated. His records showed him that there were fluctuations in the sales of the store from year to year and even from month to month. As far as the volume of business was concerned, he was fairly well satisfied that his store was securing a share of the business which was larger than that done by any similar store in the neighborhood. He also knew that year by year the sales of the store had grown. Although this general information was pleasing, he was anxious to know whether his store had maintained a favorable position over a long period of time. At any given time he knew the loss or gain in sales in relation to the previous month or the corresponding month of the preceding year, but he was not certain as to the picture which the whole series of years would present. In addition, he wanted to use the past experience as a basis for budgeting the purchase of merchandise. Consequently, he decided to make a time series analysis of his monthly sales figures. The trend, the seasonal index, and the cycle relatives were calculated as shown in Tables 27 to 30. When the trend figures were compared with similar measures for the series of bank clearings as reported by the local clearing house, it was evident that his store had made more progress than the progress which the clearings seemed to indicate for the business of the community.

A like comparison of cycle relatives indicated that the deviations from normal sales were very similar to the corresponding deviations in clearings. This suggested that the loss of business which the store suffered from time to time was, to a large extent, a reflection of local business conditions.

Aside from the December peak, the indexes of seasonal variation seemed to show deviations from the average month which were not as large as those which he suspected to be true for some other stores. The manager interpreted this as an indication that the sales of his store showed on the average smaller month-to-month fluctuations than was true of other stores.

Normal values then were computed. The manager believed that these would give him a standard which would help to prevent his store from slipping back from its hard-won position. Moreover, he expected to use the standard as a basis for forecasting

the sales. The following is an illustration of the procedure used in making the forecast:

The trend point for December, 1928, was $372,000.

The monthly growth increment was $900.

The seasonal index for January was 96.9%.

Assuming that January sales, because of the prospect of exceptionally good business, were to be 5% above normal, the January estimate of sales would be as follows:

January, 1929, trend point = $372,000 + $900 = $372,900

$$\text{January, 1929, normal} = \frac{\$372{,}900 \times 96.9}{100} = \$361{,}340.10$$

January, 1929, estimated = 105% of $361,340.10 = $379,-407.10

Other months were calculated by a similar method. The budget estimate for the first half of the year 1928 was set up. Actual sales were checked against these estimates as time went on so that a comparison was always available.

TABLE 22

Morgan Department Store

Inventories (End of Month)

Unit: $100,000

Month	1920	1921	1922	1923	1924	1925	1926	1927	1928
January	2.56	2.74	3.21	3.09	2.33	3.04	4.67	3.97	4.22
February	2.28	1.99	2.46	3.05	2.32	2.42	3.72	3.24	3.32
March	3.02	3.12	2.95	3.44	2.46	4.74	3.37	3.94	3.68
April	3.24	3.09	2.86	3.35	2.72	4.64	3.38	4.02	3.64
May	2.94	3.19	3.51	2.92	2.74	5.31	3.54	3.94	3.52
June	2.78	3.16	3.55	2.99	2.62	5.16	3.59	3.78	3.52
July	3.01	3.35	3.14	2.75	3.22	5.17	3.37	3.97	4.27
August	2.15	3.39	2.53	2.78	2.45	4.28	3.15	3.55	3.68
September	3.14	4.22	3.58	3.44	3.18	5.28	3.64	4.28	4.14
October	3.48	4.27	3.88	3.87	3.75	6.24	5.00	4.90	5.43
November	3.41	4.38	4.17	3.97	3.92	7.15	5.17	5.50	5.87
December	2.55	3.68	3.37	2.74	3.31	6.46	4.70	4.87	4.74
Total	34.56	40.58	39.21	38.39	35.02	59.89	47.30	49.96	50.03
Average	2.88	3.38	3.27	3.20	2.92	4.99	3.94	4.16	4.17

TABLE 23

Calculation of Secular Trend

Odd Number of Years; Noncumulative Data

Inventory (End of Month)

Unit: $100,000

Calculation of Annual Increment and Trend Value at Mid-date

Year	Average Annual Inventory Y	Time Deviation X	XY	X^2	Trend Y_c
1920	2.9	−4	−11.6	16	3.0
1921	3.4	−3	−10.2	9	...
1922	3.3	−2	− 6.6	4	...
1923	3.2	−1	− 3.2	1	...
1924	2.9	0	0	0	3.7
1925	5.0	+1	+ 5.0	1	...
1926	3.9	+2	+ 7.8	4	...
1927	4.2	+3	+12.6	9	...
1928	4.2	+4	+16.8	16	4.4
Total	$\Sigma Y = 33.0$		−31.6 +42.2 $\Sigma XY = +10.6$	$\Sigma X^2 = 60$	

Average $a = \dfrac{\Sigma Y}{N} = 3.667$

Annual increment $b = \dfrac{\Sigma XY}{\Sigma X^2} = \dfrac{10.6}{60} = 0.177$

Trend value of middle year, July 15, 1924 $= \dfrac{\Sigma Y}{N} = \dfrac{33.0}{9} = 3.667$

Calculation of annual trend values, illustrative examples

$$
\begin{aligned}
1920 &= 3.667 - 4 \times 0.177 = 2.959 \\
1928 &= 3.667 + 4 \times 0.177 = 4.375 \\
1935 &= 3.667 + 11 \times 0.177 = 5.614
\end{aligned}
$$

Calculation of monthly increment and trend value of a calendar month

Annual increment in trend values $= 0.177$
Monthly increment in trend values $= 0.177 \div 12 = 0.015$
Trend inventory on July 15, 1924 (mid-date) $= 3.667$

Trend inventory on July 31, 1924 $= 3.667 + \dfrac{0.015}{2} = 3.67$

Calculation of monthly trend values, illustrative examples

Trend in February, 1922 $= 3.67 - 29 \times 0.015 = 3.23$
Trend in November, 1922 $= 3.67 - 20 \times 0.015 = 3.37$
Trend in February, 1928 $= 3.67 + 43 \times 0.015 = 4.32$
Trend in November, 1928 $= 3.67 + 52 \times 0.015 = 4.45$

TABLE 24

Example of Calculation of Per Cent of Actual to Secular Trend

Inventory (End of Month)

Unit: $100,000

Date	Actual Inventory	Trend Values[1]	Per Cent of Actual to Trend
February, 1922	2.46	3.23	2.46 ÷ 3.23 = 76 %
November, 1922	4.17	3.37	4.17 ÷ 3.37 = 124%
February, 1928	3.32	4.32	3.32 ÷ 4.32 = 77 %
November, 1928	5.87	4.45	5.87 ÷ 4.45 = 132%

[1] Trend fitted to odd number of years, noncumulative data. See Table 23.

TABLE 25

Morgan Department Store

Sales

Unit: $100,000

Month	1920	1921	1922	1923	1924	1925	1926	1927	1928
January	2.22	3.78	2.74	2.76	3.09	3.42	3.44	4.04	4.15
February	1.68	2.58	2.18	2.15	2.58	2.26	2.29	3.19	3.05
March	2.58	3.58	2.75	2.95	2.09	2.19	2.58	2.69	2.48
April	2.78	3.72	3.95	3.39	3.31	3.48	3.52	4.20	3.82
May	2.82	3.67	3.77	3.51	3.54	3.16	3.52	3.31	3.61
June	3.09	3.90	3.72	3.77	3.94	4.04	3.78	3.95	3.92
July	1.80	2.05	2.43	2.61	3.26	2.25	2.56	2.49	3.09
August	1.92	2.23	2.42	2.39	2.66	1.89	2.45	3.39	3.18
September	2.08	2.15	2.53	2.25	1.79	1.92	2.23	2.66	2.71
October	2.82	3.25	3.35	3.09	2.58	3.37	3.47	3.81	3.41
November	3.11	3.45	3.38	3.19	3.09	3.94	4.04	5.60	4.64
December	4.18	4.74	4.93	5.26	5.48	6.29	5.87	6.82	6.16
Total	31.08	39.10	38.15	37.32	37.41	38.21	39.75	46.15	44.22

TABLE 26

Calculation of Secular Trend and Monthly Trend Values

Even Number of Years;[1] Cumulative Data in Noncumulative Form

Average Monthly Sales

Unit: $100,000

Calculation of Annual Increment and Trend Value at Mid-date

Year	Average Monthly Sales Y	Time Deviation $2X$	$2XY$	$4X^2$	Trend Y_c
1921	3.26	−7	−22.82	49	3.04
1922	3.18	−5	−15.90	25	....
1923	3.11	−3	− 9.33	9	....
1924	3.12	−1	− 3.12	1	....
1925	3.18	+1	+ 3.18	1	3.38
1926	3.31	+3	+ 9.93	9	....
1927	3.85	+5	+19.25	25	....
1928	3.69	+7	+25.83	49	3.63
Total.	$\Sigma Y = 26.70$		+58.19 −51.17	168	
			$\Sigma 2XY = +\ 7.02$ $\Sigma XY = +\ 3.51$	$\Sigma X^2 = \frac{168}{4} = 42$	

$$\text{Average } a = \frac{\Sigma Y}{N} = 3.3375$$

$$\text{Annual increment } b = \frac{\Sigma XY}{\Sigma X^2} = \frac{3.51}{42} = 0.0836$$

$$\text{Trend value of middle year (1924–1925)} = \frac{\Sigma Y}{N} = \frac{26.70}{8} = 3.3375$$

Calculation of annual trend values, illustrative examples

$$\begin{aligned} 1925 &= 3.3375 + \tfrac{1}{2} \times 0.0836 = 3.3793 \\ 1921 &= 3.3793 - 4 \times 0.0836 = 3.0449 \\ 1928 &= 3.3793 + 3 \times 0.0836 = 3.6301 \\ 1935 &= 3.3793 + 10 \times 0.0836 = 4.2153 \end{aligned}$$

Calculation of monthly increment and trend value

Annual increment in trend value = 0.0836
Monthly increment = 0.0836 ÷ 12 = 0.0070
Monthly sales of middle year (1924–1925) = 3.3375

$$\text{Monthly sales for January, 1925} = 3.3375 + \frac{0.0070}{2} = 3.3410$$

Calculation of monthly trend values, illustrative examples

Monthly sales, July, 1922 = 3.3410 − 30 × 0.0070 = 3.130
Monthly sales, December, 1922 = 3.3410 − 25 × 0.0070 = 3.166
Monthly sales, July, 1928 = 3.3410 + 42 × 0.0070 = 3.635
Monthly sales, December, 1928 = 3.3410 + 47 × 0.0070 = 3.670

[1] Calculation is included since it illustrates the method used when the trend is fitted to a period covering an even number of years.

TABLE 27

Calculation of Secular Trend

Odd Number of Years; Cumulative Data in Noncumulative Form

Sales

Unit: $100,000

Year	Average Monthly Sales Y	Time Deviation X	XY	X^2	Trend Y_c
1920	2.59	−4	−10.36	16	2.82
1921	3.26	−3	− 9.78	9	....
1922	3.18	−2	− 6.36	4	....
1923	3.11	−1	− 3.11	1	....
1924	3.12	0	0.0	0	3.25
1925	3.18	+1	+ 3.18	1	....
1926	3.31	+2	+ 6.62	4	....
1927	3.85	+3	+11.55	9	....
1928	3.69	+4	+14.76	16	3.68
Total.	$\Sigma Y = 29.29$		+36.11 −29.61	$X^2 = 60$	
			$\Sigma XY = + 6.50$		

$$\text{Average } a = \frac{\Sigma Y}{N} = \frac{29.29}{9} = 3.25$$

$$\text{Annual increment } b = \frac{\Sigma XY}{X^2} = \frac{6.50}{60} = 0.108$$

$$\text{Trend value at mid-date, July 1, 1924} = \frac{\Sigma Y}{N} = \frac{29.29}{9} = 3.25$$

Calculation of annual trend values, illustrative examples

$$1920 = 3.25 - 4 \times 0.108 = 2.82$$
$$1928 = 3.25 + 4 \times 0.108 = 3.68$$
$$1935 = 3.25 + 11 \times 0.108 = 4.44$$

Calculation of monthly increment and trend value at mid-date

Annual increment in trend values $= 0.108$

Monthly increment in trend values $= \dfrac{0.108}{12} = 0.009$

Monthly sales at July 1, 1924 (mid-date) $= 3.25$
Monthly sales for July, 1924 $= 3.25 + \frac{1}{2} \times 0.009 = 3.25$

Calculation of monthly trend values, illustrative examples

Monthly sales for July, 1922 $= 3.25 - 24 \times 0.009 = 3.03$
Monthly sales for December, 1922 $= 3.25 - 19 \times 0.009 = 3.08$
Monthly sales for July, 1928 $= 3.25 + 48 \times 0.009 = 3.68$
Monthly sales for December, 1928 $= 3.25 + 53 \times 0.009 = 3.73$

TABLE 28

Calculation of Secular Trend

Special Case: Cumulative Data in Cumulative Form

Sales

Unit: $100,000

Calculation of Annual Increment and Trend Value for Middle Year

Year	Annual Sales Y	Time Deviation X	XY	X^2	Trend Y_c
1920	31	−4	−124	16	34
1921	39	−3	−117	9	..
1922	38	−2	− 76	4	..
1923	37	−1	− 37	1	..
1924	37	0	0	0	39
1925	38	+1	+ 38	1	..
1926	40	+2	+ 80	4	..
1927	46	+3	+138	9	..
1928	44	+4	+176	16	44
Total	$\Sigma Y = 350$		+432 −354	$\Sigma X^2 = 60$	
			$\Sigma XY = +\ 78$		

$$\text{Average } a = \frac{\Sigma Y}{N} = 38.89$$

$$\text{Annual increment } b = \frac{\Sigma XY}{\Sigma X^2} = \frac{78}{60} = 1.30$$

$$\text{Trend value for middle year, 1924} = \frac{\Sigma Y}{N} = \frac{350}{9} = 38.89$$

Calculation of annual trend values, illustrative examples

$$1920 = 38.89 - 4 \times 1.3 = 33.69$$
$$1928 = 38.89 + 4 \times 1.3 = 44.09$$
$$1935 = 38.89 + 11 \times 1.3 = 53.19$$

Calculation of monthly increment and trend value for July of middle year

Annual increment in trend values (cumulative) = 1.30
Monthly increment in trend values = 1.30 ÷ 144 = 0.009

Trend value of annual sales for middle year, 1924 = 38.89
Trend value of monthly sales for middle month of 1924 = 38.89 ÷ 12 = 3.24
Trend value of monthly sales for July, 1924 = 3.24 + ½(.009) = 3.24

TABLE 29

Calculation of Indexes of Seasonal Variation

Link Relative Method

Morgan Department Store

Sales

Calculation of Link Relatives, and Selection of Medians

Year	Feb / Jan.	Mar. / Feb.	Apr. / Mar.	May / Apr.	June / May	July / June	Aug. / July	Sept. / Aug.	Oct. / Sept.	Nov. / Oct.	Dec. / Nov.	Jan. / Dec.
1920	76	154	108	101	110	58	107	108	136	110	134	90
1921	68	139	104	99	106	53	109	96	151	106	137	58
1922	80	126	144	95	99	65	100	104	132	101	146	56
1923	78	137	115	104	107	69	92	94	137	103	165	59
1924	83	81	158	107	111	83	82	67	144	120	177	62
1925	66	97	159	91	128	56	84	102	175	117	160	55
1926	67	113	136	100	107	68	96	91	156	116	145	69
1927	79	84	156	79	119	63	136	78	143	147	122	61
1928	73	81	154	94	109	79	103	85	126	136	133	..
Median Link Relatives	76	113	144	99	109	65	100	94	143	116	145	60

Calculation of the Indexes

Month	Median Link Relatives	Chain Relatives	Correcting Factor	Corrected Chain Relatives	Index
		100.0	.....	100.0	96.90
F/J	76	76.0	101.5	74.9	72.60
M/F	113	85.9	103.0	83.4	80.85
A/M	144	123.7	104.4	118.5	114.90
M/A	99	122.5	105.9	115 7	112.20
J/M	109	133.5	107.4	124.3	120.50
J/J	65	86.8	108.9	79.7	77.30
A/J	100	86.8	110.4	78.6	76.20
S/A	94	81.6	111.8	73.0	70.80
O/S	143	116.7	113.3	103.0	99.85
N/O	116	135.4	114.8	117.9	114.30
D/N	145	196.3	116.3	168.8	163.60
J/D	60	117.8	117.8	......	....
				1,237.8	1,200.00

Monthly adjustment = 17.8 ÷ 12 = 1.48.

1,237.8 ÷ 12 = 103.15 = New Base.

TABLE 30

Calculation of Cycle Relatives
Morgan Department Store
Sales
Unit: $100.000

Date	Actual Data A	Trend Values T	Actual as Per Cent of Trend A/T	Seasonal Index S	Cycle Relatives—Per Cent of Actual to Normal $C = A/TS$	Five Months' Moving Average of Cycle Relatives
1920						
January	2.22	2.76	80.4	96.90	83.0	....
February	1.68	2.77	60.6	72.60	83.5	....
March	2.58	2.78	92.8	80.85	114.8	91.6
April	2.78	2.79	99.6	114.90	86.7	93.2
May	2.82	2.80	100.7	112.20	89.8	93.0
June	3.09	2.81	110.0	120.50	91.3	87.9
July	1.80	2.82	63.8	77.30	82.5	91.2
August	1.92	2.83	67.8	76.20	89.0	93.1
September	2.08	2.84	73.2	70.80	103.4	94.0
October	2.82	2.84	99.3	99.85	99.4	95.3
November	3.11	2.85	109.1	114.30	95.5	104.7
December	4.18	2.86	146.2	163.60	89.4	108.7
1921						
January	3.78	2.87	131.7	96.90	135.9	119.5
February	2.58	2.88	89.6	72.60	123.4	122.7
March	3.58	2.89	123.9	80.85	153.2	127.4
April	3.72	2.90	128.3	114.90	111.7	122.5
May	3.67	2.90	126.6	112.20	112.8	115.9
June	3.90	2.91	134.0	120.50	111.2	105.3
July	2.05	2.92	70.2	77.30	90.8	103.6
August	2.23	2.93	76.1	76.20	99.9	103.1
September	2.15	2.94	73.1	70.80	103.2	101.3
October	3.25	2.95	110.2	99.85	110.4	102.6
November	3.45	2.96	116.6	114.30	102.0	101.6
December	4.74	2.97	159.6	163.60	97.6	101.0
1922						
January	2.74	2.98	91.9	96.90	94.8	101.6
February	2.18	2.99	72.9	72.60	100.4	104.2
March	2.75	3.00	91.7	80.85	113.4	107.0
April	3.95	3.00	131.7	114.90	114.6	108.4
May	3.77	3.01	125.2	112.20	111.6	109.2
June	3.72	3.02	123.2	120.50	102.2	107.5
July	2.43	3.02	80.5	77.30	104.1	108.1
August	2.42	3.03	79.9	76.20	104.9	107.7
September	2.53	3.04	83.2	70.80	117.5	106.6
October	3.35	3.05	109.8	99.85	110.0	105.5
November	3.38	3.06	110.5	114.30	96.7	103.0
December	4.93	3.07	160.6	163.60	98.2	98.7
1923						
January	2.76	3.08	89.6	96.90	92.5	100.2
February	2.15	3.09	69.6	72.60	95.9	99.8
March	2.95	3.10	95.2	80.85	117.7	100.3
April	3.39	3.11	109.0	114.90	94.9	101.9
May	3.51	3.11	112.9	112.20	100.6	104.3
June	3.77	3.12	120.8	120.50	100.2	100.7
July	2.61	3.13	83.4	77.30	107.9	101.9
August	2.39	3.14	76.1	76.20	99.9	101.3
September	2.25	3.15	71.4	70.80	100.8	98.9
October	3.09	3.16	97.8	99.85	97.9	97.5
November	3.19	3.17	100.6	114.30	88.0	97.6
December	5.26	3.18	165.4	163.60	101.1	99.6
1924						
January	3.09	3.19	96.9	96.90	100.0	96.2
February	2.58	3.20	80.6	72.60	111.0	96.5
March	2.09	3.20	65.3	80.85	80.8	95.9
April	3.31	3.21	103.1	114.90	89.7	96.1
May	3.54	3.22	109.9	112.20	98.0	100.0
June	3.94	3.23	122.0	120.50	101.2	105.3
July	3.26	3.24	100.6	77.30	130.1	102.8

TABLE 30 (*Continued*)

Date	Actual Data A	Trend Values T	Actual as Per Cent of Trend A/T	Seasonal Index S	Cycle Relatives, Per Cent of Actual to Normal $C = A/TS$	Five Months' Moving Average of Cycle Relatives
1924 (*Cont.*)						
August	2.66	3.25	81.8	76.20	107.3	99.0
September	1.79	3.26	54.9	70.80	77.5	95.3
October	2.58	3.27	78.9	99.85	79.0	89.6
November	3.09	3.28	94.2	114.30	82.4	89.6
December	5.48	3.29	166.6	163.60	101.8	93.0
1925						
January	3.42	3.29	104.0	96.90	107.3	93.6
February	2.26	3.30	68.5	72.60	94.4	95.3
March	2.19	3.31	66.2	80.85	81.9	91.9
April	3.48	3.32	104.8	114.90	91.2	90.5
May	3.16	3.33	94.9	112.20	84.6	89.0
June	4.04	3.34	121.0	120.50	100.4	87.4
July	2.25	3.35	67.2	77.30	86.9	85.3
August	1.89	3.36	56.3	76.20	73.8	88.3
September	1.92	3.37	57.0	70.80	80.5	88.6
October	3.37	3.38	99.7	99.85	99.8	93.9
November	3.94	3.38	116.6	114.30	102.0	100.0
December	6.29	3.39	185.5	163.60	113.4	102.4
1926						
January	3.44	3.40	101.2	96.90	104.4	101.1
February	2.29	3.41	67.2	72.60	92.6	98.6
March	2.58	3.42	75.4	80.85	93.3	94.2
April	3.52	3.43	102.6	114.90	89.3	91.5
May	3.52	3.44	102.3	112.20	91.2	92.1
June	3.78	3.45	109.6	120.50	91.0	92.0
July	2.56	3.46	74.0	77.30	95.7	92.3
August	2.45	3.47	70.6	76.20	92.7	94.0
September	2.23	3.47	64.3	70.80	90.8	96.1
October	3.47	3.48	99.7	99.85	99.9	97.4
November	4.04	3.49	115.8	114.30	101.3	102.7
December	5.87	3.50	167.7	163.60	102.5	109.5
1927						
January	4.04	3.51	115.1	96.90	118.8	108.3
February	3.19	3.52	90.6	72.60	124.8	108.7
March	2.69	3.53	76.2	80.85	94.2	104.8
April	4.20	3.54	118.6	114.90	103.2	99.5
May	3.31	3.55	93.2	112.20	83.1	92.6
June	3.95	3.56	111.0	120.50	92.1	98.7
July	2.49	3.56	69.9	77.30	90.4	99.0
August	3.39	3.57	95.0	76.20	124.7	103.7
September	2.66	3.58	74.3	70.80	104.9	112.5
October	3.81	3.59	106.1	99.85	106.2	117.5
November	5.60	3.60	155.6	114.30	136.1	116.2
December	6.82	3.61	188.9	163.60	115.5	118.4
1928						
January	4.15	3.62	114.6	96.90	118.3	114.0
February	3.05	3.63	84.0	72.60	115.7	105.0
March	2.48	3.64	68.1	80.85	84.2	99.5
April	3.82	3.65	104.7	114.90	91.1	93.6
May	3.61	3.65	98.9	112.20	88.1	92.2
June	3.92	3.66	107.1	120.50	88.9	98.1
July	3.09	3.67	84.2	77.30	108.9	100.6
August	3.18	3.68	86.4	76.20	113.4	101.4
September	2.71	3.69	73.4	70.80	103.7	105.5
October	3.41	3.70	92.2	99.85	92.3	104.0
November	4.64	3.71	125.1	114.30	109.4	.....
December	6.16	3.72	165.6	163.60	101.2	.....

TABLE 31
Morgan Department Store
Calculation of Indexes of Seasonal Variation
(Moving Average Method)
Sales
Unit: $100,000

Calculation of Moving Totals

Year	Jan.	Feb.	Mar.	Apr.	May	June	July	Aug.	Sept.	Oct.	Nov.	Dec.
1920	.....	.....	.....	.....	.....	.....	31.08	32.64	33.54	34.54	35.48	36.33
1921	37.14	37.39	37.70	37.77	38.20	38.54	39.10	38.06	37.66	36.83	37.06	37.16
1922	36.98	37.36	37.55	37.93	38.03	37.96	38.15	38.17	38.14	38.34	37.78	37.52
1923	37.57	37.75	37.72	37.44	37.18	36.99	37.32	37.65	38.08	37.22	37.14	37.17
1924	37.34	37.99	38.26	37.80	37.29	37.19	37.41	37.74	37.42	37.52	37.69	37.31
1925	37.41	36.40	35.63	35.76	36.55	37.40	38.21	38.23	38.26	38.65	38.69	39.05
1926	38.79	39.10	39.66	39.97	40.07	40.17	39.75	40.35	41.25	41.36	42.04	41.83
1927	42.00	41.93	42.87	43.30	43.64	45.20	46.15	46.26	46.12	45.91	45.53	45.83
1928	45.80	46.40	46.19	46.24	45.84	44.88	44.22	.....	.....	.....	.....	.....

Calculation of Moving Averages

Year	Jan.	Feb.	Mar.	Apr.	May	June	July	Aug.	Sept.	Oct.	Nov.	Dec.
1920	....	....	....	....	....	....	2.59	2.72	2.80	2.88	2.96	3.03
1921	3.10	3.12	3.14	3.15	3.18	3.21	3.26	3.17	3.14	3.07	3.09	3.10
1922	3.08	3.11	3.13	3.16	3.17	3.16	3.18	3.18	3.18	3.20	3.15	3.13
1923	3.13	3.15	3.14	3.12	3.10	3.08	3.11	3.14	3.17	3.10	3.10	3.10
1924	3.11	3.17	3.19	3.15	3.11	3.10	3.12	3.15	3.12	3.13	3.14	3.11
1925	3.12	3.03	2.97	2.98	3.05	3.12	3.18	3.19	3.19	3.22	3.22	3.25
1926	3.23	3.26	3.31	3.33	3.34	3.35	3.31	3.36	3.44	3.45	3.50	3.49
1927	3.50	3.49	3.57	3.61	3.64	3.77	3.85	3.86	3.84	3.83	3.79	3.82
1928	3.82	3.87	3.85	3.85	3.82	3.74	3.69	....	....	....	....	....

TABLE 31 (*Continued*)

Calculation of Relatives of Original Data to Moving Averages

Year	Jan.	Feb.	Mar.	Apr.	May	June	July	Aug.	Sept.	Oct.	Nov.	Dec.
1920	......	....	......	.....	.....	.....	69.5	70.6	74.3	97.9	105.1	138.0
1921	121.9	82.7	114.0	118.1	115.4	121.5	62.9	70.3	68.5	105.9	111.7	152.9
1922	89.0	70.1	87.9	125.0	118.9	117.7	76.4	76.1	79.6	104.7	107.3	157.5
1923	88.2	68.3	93.9	108.7	113.2	122.4	83.9	76.1	71.0	99.7	102.9	170.0
1924	99.4	81.4	65.5	105.1	113.8	127.1	104.5	84.4	57.4	82.4	98.4	176.2
1925	109.6	74.6	73.7	116.8	103.6	129.5	70.8	59.2	60.2	104.7	122.4	193.5
1926	106.5	70.2	77.9	105.7	105.4	112.8	77.3	72.9	64.8	100.6	115.4	168.2
1927	115.4	91.4	75.4	116.3	90.9	104.8	64.7	87.8	69.3	99.5	147.8	178.5
1928	108.6	78.8	64.4	99.2	94.5	104.8	83.7	....	....	......	.....	.....
Median Relatives	107.55	76.7	76.65	112.5	109.3	119.6	70.8	74.5	68.9	100.15	109.5	169.1
Seasonal Index..	108.00	77.0	77.00	112.9	109.7	120.1	71.1	74.8	69.2	100.50	109.9	169.8

Total median relatives 1,195.25
Total seasonal index 1,200.0

TABLE 32
Morgan Department Store
Calculation of Indexes of Seasonal Variation
(Per Cent of Trend Method)
Sales
Unit $100,000
Relatives of Original Data to Trend Values

Year	Jan.	Feb.	Mar.	Apr.	May	June	July	Aug.	Sept.	Oct.	Nov.	Dec.
1920	80.4	60.6	92.8	99.6	100.7	110.0	63.8	67.8	73.2	99.3	109.1	146.2
1921	131.7	89.6	123.9	128.3	126.6	134.0	70.2	76.1	73.1	110.2	116.6	159.6
1922	91.9	72.9	91.7	131.7	125.2	123.2	80.5	79.9	83.2	109.8	110.4	160.6
1923	89.6	69.6	95.2	109.0	112.9	120.8	83.4	76.1	71.4	97.8	100.6	165.4
1924	96.9	80.6	65.3	103.1	109.9	122.0	100.6	81.8	54.9	78.9	94.2	166.6
1925	104.0	68.5	66.2	104.8	94.9	121.0	67.2	56.3	57.0	99.7	116.6	185.5
1926	101.2	67.2	75.4	102.6	102.3	109.6	74.0	70.6	64.3	99.7	115.8	167.7
1927	115.1	90.6	76.2	118.6	93.2	111.0	69.9	95.0	74.3	106.1	155.6	188.9
1928	114.6	84.0	68.1	104.7	98.9	107.1	84.2	86.4	73.4	92.2	125.1	165.6
Ave. of Middle Three.	100.7	74.4	81.1	106.17	104.3	117.6	74.9	77.4	72.6	99.6	114.3	165.9
Seasonal Index	101.7	75.1	81.9	107.20	105.3	118.6	75.6	78.1	73.3	100.5	115.4	167.4

Total of average of middle three 1,188.97
Total seasonal index 1,200.00

INDIAN ABRASIVE COMPANY

Problem Involving Calculation of Compound Interest Trend

The Indian Abrasive Company manufactured grindstones and abrasive wheels which were useful for such things as metal and stone polishing, and sharpening tools. Inasmuch as practically all of their output was sold to other manufacturing industries, it was felt that their business would fluctuate synchronously with the volume of manufacture in the United States. To ascertain to what extent this was the case, the executive committee suggested that a study be made of the nature of the relationship. They thought it would be helpful in planning production if a way were discovered to forecast demand for abrasive products from an index of manufacturing activity.

The figures given below present the annual sales of Indian Abrasive Company abrasives in tons along with the Index of Production Activity constructed by the Federal Reserve Bank of New York.

As the Federal Reserve Bank Index is expressed in percentages of a calculated normal, the first step necessary before any comparison could be made was to put the abrasive sales on a comparable basis. Since the figures are given in annual form, and, therefore, are not affected by seasonal variation, "normal" in this case means "trend."

In order to determine the kind of trend line to fit, the statistician plotted the sales on both arithmetic and ratio grids. It appeared that the data on the arithmetic grid did not move along the straight line which he sketched in by observation. Therefore an equation of the type, $Y = a + bX$, would not be satisfactory for calculating trend.

On the other hand, a straight line through the data on the semilogarithmic paper seemed to be quite satisfactory, which led him to the decision to calculate a compound interest curve of the type, $\log Y = A + BX$ (see Table 33) where A is the logarithm of the intercept, and B is the logarithm of the ratio of increase in the trend.

Monthly trend values, if desired, may be found by applying to the logarithms calculations similar to those used in Table 28.

Table 34 shows the calculation of the "per cent of trend," and gives the figures for the Index of Production Activity for comparison.

TABLE 33

Fitting a Compound Interest Curve

Indian Abrasive Company

Years	X Time in Years, Measured from 1924	Y Actual Abrasive Sales in Tons	Log Y Logarithms of Corresponding Y Values	X^2	X Log Y	Log Y Computed	Trend Values Computed
1919	−5	320	2.50515	25	−12.52575	2.47457	298
1920	−4	370	2.56820	16	−10.27280	2.53818	345
1921	−3	352	2.54654	9	− 7.63962	2.60179	400
1922	−2	439	2.64246	4	− 5.28492	2.66541	463
1923	−1	578	2.76193	1	− 2.76193	2.72902	536
1924	0	560	2.74819	0	0.00000	2.79263	620
1925	+1	763	2.88252	1	2.88252	2.85624	718
1926	+2	862	2.93551	4	5.87102	2.91985	831
1927	+3	864	2.93651	9	8.80953	2.98346	963
1928	+4	1,098	3.04060	16	12.16240	3.04708	1,115
1929	+5	1,417	3.15137	25	15.75685	3.11069	1,290
	Total 0		30.71898	110	45.48232	30.71893	
					−38.48502		
					+ 6.99730		

$$B = \frac{\Sigma X \log Y}{\Sigma X^2} = \frac{6.99730}{110} = 0.063612 (= \log b)$$

$$A = \frac{\Sigma \log Y}{N} = \frac{30.71898}{11} = 2.79263 (= \log a)$$

TABLE 34

Sales and Index of Production Activity

Indian Abrasive Company

Year	Actual Sales in Tons	Trend	Per Cent of Trend	Index of Production Activity Federal Reserve Bank of N. Y.[1]
1919	320	298	107	105
1920	370	345	107	99
1921	352	400	88	85
1922	439	463	95	99
1923	578	536	108	109
1924	560	620	90	103
1925	763	718	106	110
1926	862	831	104	111
1927	864	963	90	105
1928	1,098	1,115	98	107
1929	1,417	1,290	110	104

[1] Represents volume of activity in producers' goods, consumers' goods, employment, motor vehicles, and building contracts. The calculated normal equals 100%.

TABLE 35

Steel Ingot Production, United States; Fitting of the Line of Trend

Logarithmic Curve, $\log Y = a + bX + cX^2$

(In Thousands of Gross Tons)

Years	Five Years' Total	Five Years' Annual Average Y	$\log Y$	X	$X \log Y$	$X^2 \log Y$	bX	cX^2	Calculated $\log Y$	Calculated Y
1883–1887	10,513	2,103	3.32284	−4	−13.29136	53.16544	−0.66940	−0.25824	3.29089	1,954
1888–1892	18,812	3,762	3.57542	−3	−10.72626	32.17878	−0.50205	−0.14526	3.57122	3,726
1893–1897	26,177	5,235	3.71892	−2	− 7.43784	14.87568	−0.33470	−0.06456	3.81927	6,596
1898–1902	56,968	11,394	4.05668	−1	− 4.05668	4.05668	−0.16735	−0.01614	4.03504	10,840
1903–1907	92,286	18,457	4.26616	0	....	....	....	....	4.21853	16,540
1908–1912	115,444	23,089	4.36340	+1	4.36340	4.36340	0.16735	−0.01614	4.36974	23,428
1913–1917	168,405	33,681	4.52738	+2	9.05476	18.10952	0.33470	−0.06456	4.48867	30,809
1918–1922	171,419	34,284	4.53509	+3	13.60527	40.81581	0.50205	−0.14526	4.57532	37,612
1923–1927	214,500	42,900	4.63246	+4	18.52984	74.11936	0.66940	−0.25824	4.62969	42,628
							0.83675	−0.40350	4.65178	44,852
			36.99834		45.55327	241.68467	1.00410	−0.58104	4.64159	43,812
					−35.51214	....				
							1.17145	−0.79086	4.59912	39,730
					10.04113	....	1.33880	−1.03296	4.52437	33,448

$N = 9$

$\Sigma X^2 = 60$

$\Sigma X^4 = 708$

$\Sigma(\log Y) = Na + c\Sigma X^2$

$\Sigma(X \log Y) = b\Sigma X^2$

$\Sigma(X^2 \log Y) = a\Sigma X^2 + c\Sigma X^4$

Substituting the computed values, these become

$$36.99834 = 9a + 60c \quad (1)$$
$$10.04113 = 60b \quad (2)$$
$$241.68466 = 60a + 708c \quad (3)$$

Divide (1) by 3 and (3) by 20

$$12.33278 = 3a + 20c$$
$$12.08423 = 3a + 35.4c$$
$$0.24855 = -15.4c$$
$$-0.01614 = c$$

Substitute in (1)

$$36.99834 = 9a - 0.9684$$
$$4.21853 = a$$

From (2)

$$0.16735 = b$$

TABLE 36

Steel Ingot Production, United States

Fitting of the Line of Trend

Gompertz Curve, $\log Y = \log a + c^X \log b$

Year	X	Y	$\log Y$		
1885	0	2,103	3.32284		
1890	1	3,762	3.57542	10.61718	
1895	2	5,235	3.71892	(S_1)	
					2.06906 ($S_2 - S_1$)
1900	3	11,394	4.05668		
1905	4	18,457	4.26616	12.68624	
1910	5	23,089	4.36340	(S_2)	
					1.00869 ($S_3 - S_2$)
1915	6	33,681	4.52738		
1920	7	34,284	4.53509	13.69493	
1925	8	42,900	4.63246	(S_3)	

$$c^3 = \frac{1.00869}{2.06906} = 0.48751 \qquad c^3 = 0.48751$$

$$\begin{aligned}
\log c &= \tfrac{1}{3} \log 0.48751 = \tfrac{1}{3}(9.68798 - 10) = 9.89599 - 10 \\
c &= 0.78703 \\
c^3 &= 0.48751 \\
c^3 - 1 &= -0.51249 \\
(c^3 - 1)^2 &= 0.26265 \\
c - 1 &= -0.21297
\end{aligned}$$

$$\log b = (S_2 - S_1)\frac{c - 1}{(c^3 - 1)^2} = -2.06906\frac{0.21297}{0.26265} = -\frac{0.44065}{0.26265} = -1.67771$$

$$\begin{aligned}
\log a &= \tfrac{1}{3}\left(S_1 - \frac{c^3 - 1}{c - 1}\log b\right) = \tfrac{1}{3}\left(10.61718 - \frac{0.51249}{0.21297}\log b\right) \\
&= \tfrac{1}{3}\left(10.61718 + \frac{0.85981}{0.21297}\right) = \tfrac{1}{3}(10.61718 + 4.03724) \\
&= \tfrac{1}{3}(14.65442) = 4.88481
\end{aligned}$$

TABLE 36 (*Continued*)

Steel Ingot Production, United States

Fitting of the Line of Trend

Gompertz Curve, $\log y = \log a + c^x \log b$

$(\log b = -1.67771)$ $(\log a = 4.88481)$

X	ϵ^X	$c^X \log b$	$\log Y$	Calculated Y	
0	1.00000	−1.67771	3.20710	1,611	
1	0.78703	−1.32041	3.56440	3,668	
2	0.61942	−1.03921	3.84560	7,008	
3	0.48750	−0.81788	4.06693	11,666	
4	0.38368	−0.64370	4.24111	17,423	
5	0.30197	−0.50662	4.37819	23,889	
6	0.23766	−0.39872	4.48609	30,626	
7	0.18705	−0.31382	4.57099	37,239	
8	0.14721	−0.24698	4.63783	43,434	
9	0.11586	−0.19438	4.69043	49,026	Projected trend
10	0.09119	−0.15299	4.73182	53,929	
11	0.07177	−0.12041	4.76440	58,130	
12	0.05649	−0.09477	4.79004	61,665	
13	0.04446	−0.07459	4.81022	64,599	

CHAPTER IX

CORRELATION

The term "correlation" has become so common that there is danger that its underlying meaning may be forgotten. For many, the term "covariation" carries far more meaning concerning the problem, since this term implies that one quantity changes or varies when the other changes. By the word "change" is meant a change from some original value. It is customary to speak of the determining quantity as the independent variable and the other quantity as the dependent variable. The word "dependent" indicates that the second variable depends upon the value of the independent variable. Thus, if we assume that bond yields tend to increase ½ of 1% for each 1% increase of long-time money rates, the value for money rates as given at any one time is considered as the independent variable, while bond yields, which tend to follow, are considered as the dependent variable. In other words, bond yields and money rates are covariant. There exists a correlation between them.

In the above statement it will be noticed that the word "tends" was used. This is to indicate that the relationship is not mathematically exact. Thus, in our illustration that, if money rates change by exactly 1%, it is not to be expected that inevitably bond yields should change by exactly ½ of 1%, no more and no less. Bond yields will tend to change by ½ of 1%, sometimes perhaps a little more and sometimes a little less.

The statements just made also imply a trend relationship. Thus, there seems to be a tendency for a pro rata change of bond yields in relation to the price of money.

There is an additional idea contained in the statements made. The word "tends" implies not only the existence of a line of relationship, but also implies a degree of relationship for a particular correlation set up in comparison with that for other series of data. Thus, bond yields might tend to follow money rates *more closely* than some other quantity would tend to follow its corresponding variable.

There are, then, two fundamental ideas used in connection with the correlation of two variables; first, the line of the relationship which exists, and second, the degree of closeness of that relationship. More formally, these two ideas may be represented respectively by (1) the line of relationship, or of regression, and (2) the coefficient of correlation.

If we are given two series in which the figures of one are paired with those of the other, it is possible to calculate the line of relationship and the degree of correlation between the two series. When there are economic or other logical reasons for believing that a relationship actually exists, the procedure of correlation is justified, and the results are likely to prove to be of real value in the solution of business problems. Through misunderstanding of this principle, the procedure of correlation is used as a substitute for logic in an attempt to prove the existence of an economic relationship which actually does not exist. In such cases, though the degree of correlation may be high, these instances can be classed only as coincidents.

One of the simplest cases of such false procedure is the correlation of two unrelated series in both of which a steep upward trend has not been eliminated. Here the apparent correlation may be explained by the coincidence in the trends.

LINE OF RELATIONSHIP

The relationship of the dependent quantity to the independent quantity may be either direct or inverse. In the case of bond yields and money rates, the relationship is direct, because bond yields tend to increase as money rates increase. On the other hand, bond prices and money rates have an inverse relationship. This is because bond prices vary inversely to bond yields.

The line of relationship may be determined graphically, or a curve may be selected and then fitted mathematically. The graphical process usually is a purely visual one, but it is highly desirable for many business problems. Its principal uses are in those cases where only a rough estimate is desired. By this method a straight or curvilinear line is drawn to represent the arrangement of plotted points on a dot graph (see Chapter XII of Book I). As in the case of trend lines discussed above, the selection of the type of line to a large extent determines the answer. In the mathematical process, the mathematics only adjusts the line to a position which may be regarded for our

purposes as the most satisfactory. The only method to be considered here of fitting the straight or curved line will be the least squares method.

In order to show how correlation may be used to solve business problems, direct and indirect labor costs for the Ray Radio Company[1] have been correlated. The position of any straight line on any graph can be determined if two points are known. One of the two points which is most convenient to select is the point of mean, or average of the X and of the Y-values of the original data. In other words, the Y-value of this average point is the average of all of the Y-values, while the X-value is the average of all of the X-values. Thus the two coordinates of the first point on the line representing the correlation would be

$$\text{Average of } X\text{-values} = \frac{\Sigma X}{N} = 145.3^{2}$$

$$\text{Average of } Y\text{-values} = \frac{\Sigma Y}{N} = 93.7^{2}$$

The position of this point is first marked out on the graph (see (Fig. 46).

The second point can be determined from the equation of the straight line. The method of determining this equation from the given data will be described below. Assuming for the moment that it has been determined, we can find the second point easily. The equation for the correlation of indirect and direct labor for the Ray Radio Company, as given below, is $y = +0.287x$. It will be noted that a small x and a small y are used in this equation. The purpose of doing this is to indicate that the line has as its origin, or zero point, the point of average X and average Y. This is the origin or starting point from which we measure the values of x and y. If now we assume $x = 500$, then we measure 500 units to the right of the average X. The total value for X then will be *Average* $X + x = 1{,}453 + 500 = 1{,}953$. The value of Y, the dependent variable, corresponding to this final value can be determined in two steps. From the equation, y is determined as 0.287 times 500, or 144. This value should be added to the starting value, or average Y. From this we shall get *Average* $Y + y = 937 + 144 = 1{,}081$. These two values for X and Y can be plotted on the graph as shown in Fig. 46. They

[1] See Tables 37 through 40.
[2] Zeros in original data omitted.

determine a second point so that now the straight line can be drawn through these two points as shown.

The formula used to calculate the equation of the line of trend or average relationship is

$$y = \frac{\Sigma XY - \frac{\Sigma X \cdot \Sigma Y}{N}}{\Sigma X^2 - \frac{(\Sigma X)^2}{N}} x$$

In this formula the X and Y refer to the values given in the original data. The calculations using this formula are shown on page 203.

COEFFICIENT OF CORRELATION

A determination of this measure is of *secondary* importance in most business problems. In those cases in which it is of significance, it is used in comparison with another coefficient of correlation from a corresponding set of figures. When used in this way, it determines which of the two relationships is the better.

The values of the coefficient of correlation range from -1 to $+1$. The negative values indicate an inverse correlation, while the positive values indicate a direct relationship. The value unity either -1 or $+1$ means perfect inverse or direct correlation, while zero means absolutely no correlation as will be shown. In a very inexact and broad sense, a coefficient of correlation of ± 0.9 means a relatively high degree of correlation, while ± 0.5 usually indicates a more or less slight degree. It should be noted again, however, that whether a coefficient of 0.9 or 0.5 is significant in connection with any given problem depends upon the economic conditions within that problem. As a number in itself the coefficient has very little meaning.

The Pearsonian formula for the coefficient of correlation is as follows:

$$r = \pm \sqrt{1 - \frac{S_y^2}{\sigma_y^2}}$$

Here r stands for the coefficient of correlation, S_y for the standard error, and σ_y for the standard deviation. The subscript y signifies that the deviations used in calculating the standard error and the standard deviation are measured parallel to the Y or vertical axis. Obviously, the value of r is influenced by the relative value of S and σ. The significance of these two values will now be discussed.

A distinction between standard error (deviations measured from the sloping line or line of relationship) and standard deviation (deviations measured from the horizontal line) is illustrated in the following diagram.

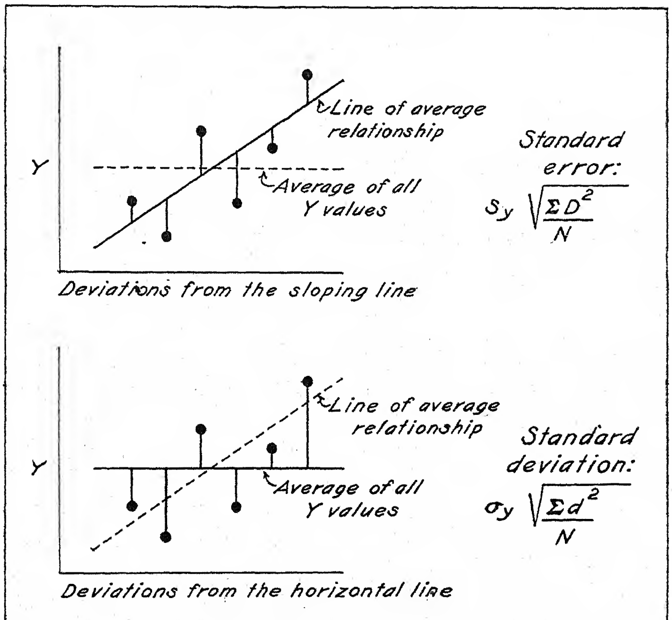


FIG. 45.

If it is desired to calculate the values of S_y and σ_y, where the values of deviations are not available, it may be done from the following formulae expressed in the original values of X and Y:

$$S_y^2 = \frac{1}{N}\left\{\Sigma Y^2 - \frac{(\Sigma Y)^2}{N} - \frac{\left[\Sigma XY - \frac{\Sigma X \Sigma Y}{N}\right]^2}{\Sigma X^2 - \frac{(\Sigma X)^2}{N}}\right\}$$

$$\sigma_y^2 = \frac{\Sigma Y^2}{N} - \left(\frac{\Sigma Y}{N}\right)^2$$

The standard error measures the closeness with which the plotted points cluster about the line of relationship, so that if the points are very close to the line of relationship, S_y is small.

The standard error in relation to the sloping line in correlation has the same significance as a standard deviation has in relation to the arithmetic average. As already explained on page 170 in connection with cyclical deviations, the deviations from the line of relationship also may be considered in the form of a regular frequency distribution. A principle borrowed from a normal curve often is found useful in modifying the estimates made from the line of relationship. The modification may be made graphically by drawing a line on each side of the line of relationship parallel to the latter and located at the vertical distance of one standard error from the latter. The belt, or "zone of estimate," formed by these two parallel lines will contain 68% of all the points of observation, provided the deviations form a normal frequency distribution. In practice it is assumed that the distribution tends to be normal with a large number of items, and the "zone of estimate" is used as a rough measure of concentration. This is illustrated in Fig. 46. Ninety-five per cent of the items are enclosed between two lines separated from the middle line by two standard errors. Ninety-nine per cent of the items are enclosed between the lines removed by three standard errors from the line of relationship. When the variation in Y-values is great it may be desirable to make the width of a zone proportionate to the heights of the corresponding Y-values, that is, wider toward the right. This can be done by expressing the standard error in per cent of the average Y.

In an extreme case, when the points are mathematically on the line of relationship, the deviations from the line are 0, so that the standard error S_y is 0. Hence, S_y^2/σ_y^2 is 0, if σ_y^2 is not 0. This makes the coefficient of correlation equal to ± 1. This extreme condition is that of perfect correlation.

In an opposite case, where, for example, the dots are scattered symmetrically above and below a horizontal line, the line of relationship fitted by the method of least squares as described in this chapter is parallel to the X-axis. For every point above the horizontal line through the center of the dots there is a symmetrically situated point below the line. In this case S_y and σ_y are both measured from the dots to the same horizontal line; consequently, they are equal. The quotient S_y/σ_y, therefore, is unity and the value of the coefficient of correlation is 0.

Pearson's formula for the coefficient of correlation, used up to this point, although simple in appearance, is not convenient

to use in connection with many examples. Ayres' formula, consequently, is used when a straight line of relationship is fitted by the method of least squares. This formula is as follows:

$$r = \frac{\Sigma XY - \dfrac{\Sigma X \cdot \Sigma Y}{N}}{\sqrt{\Sigma X^2 - \dfrac{(\Sigma X)^2}{N}}\sqrt{\Sigma Y^2 - \dfrac{(\Sigma Y)^2}{N}}}$$

This formula can be derived from Pearson's formula by a process of algebra. Proof, however, is omitted.

RAY RADIO COMPANY

CORRELATION

In July, 1929 the general manager of the Ray Radio Company requested the statistician to devise a simple method of estimating the amounts of the indirect labor pay roll which should correspond to the fluctuating amount of business of the company. The direct labor pay roll fluctuated directly with the amount of work on hand in the shop. Indirect labor pay roll, however, was more difficult to estimate. Although clerical jobs, supervision, and various other duties performed by the indirect labor force called for additional men when the amount of work increased, or required some lay-offs when shop activities slowed down, the increase or decrease was not in direct proportion to the amount of work as in the case of the direct pay roll. Consequently, the manager desired some simple method of determining excesses in the indirect labor force, so that cost for this part of the work might be controlled.

It seemed to the statistician that there was a good reason for a definite relationship between the amount of direct and the amount of indirect labor pay roll.

He calculated a straight line of relationship by the method of least squares and plotted it on a graph (see Fig. 46). He also calculated the standard error. The graph together with the figures was sent to the general manager who used it in the following way. When he received a weekly report giving the pay rolls for direct and indirect labor, he located the amount of direct pay roll on the X (or horizontal) scale of the graph, and drew a vertical line parallel to Y-axis through the point. The point of intersection of this vertical line with the line of relationship gave him a certain value on the Y (or vertical) axis which was determined by drawing a line parallel to the X-axis and reading off the scale value of the point of intersection with the Y-axis. The value so determined was the amount of the typical or average indirect pay roll corresponding to a particular value of direct pay roll. If the reported indirect figure was much larger than the value indicated on the graph, the manager asked the shop superintendent for an explanation.

For example, in a certain week the direct reported pay roll was $1,500 and the indirect was $1,000. From the diagram it was found that $1,500 on the X-scale corresponded to about $950 on the Y-scale. The reported indirect figure exceeded the

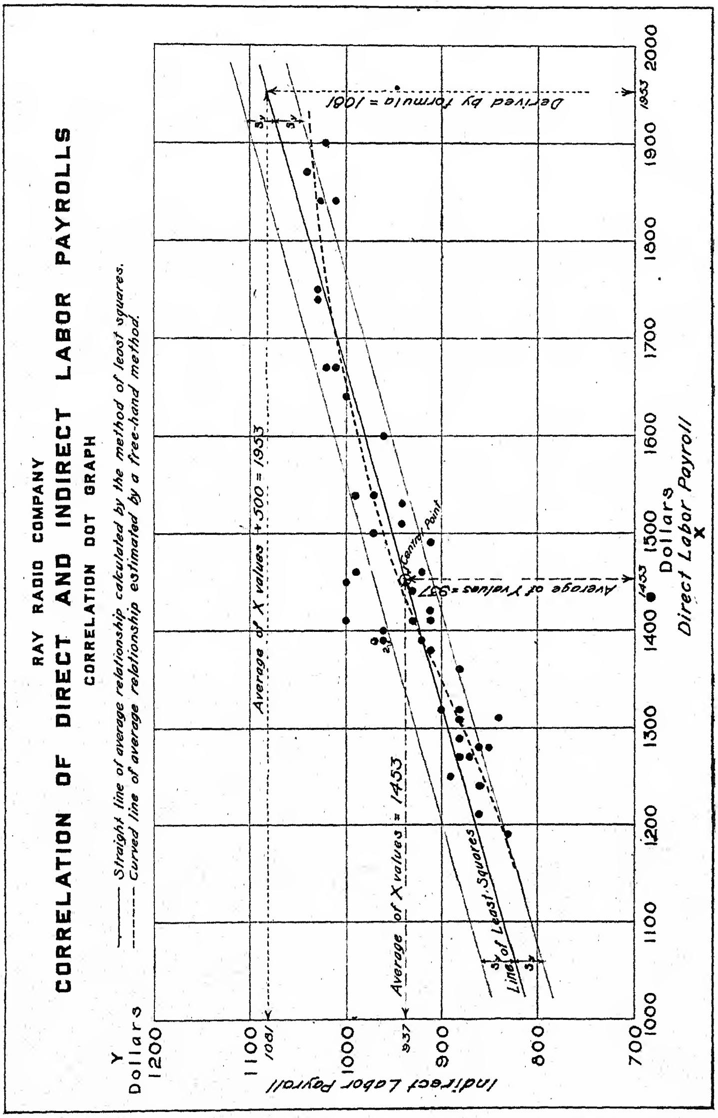

RAY RADIO COMPANY
CORRELATION OF DIRECT AND INDIRECT LABOR PAYROLLS
CORRELATION DOT GRAPH
Straight line of average relationship calculated by the method of least squares.
Curved line of average relationship estimated by a free-hand method.
Y
Dollars
Indirect Labor Payroll
Direct Labor Payroll
X
Dollars
Average of X values + 500 = 1953
Average of X values = 1453
Average of Y values = 937
Derived by formula = 1081
Central Point
Line of Least Squares
1200
1100
1081
1000
937
900
800
700
1000
1100
1200
1300
1400
1453
1500
1600
1700
1800
1900
1953
2000

FIG. 46.

estimate by \$50. Since the standard error was only \$29, the deviation of \$50 was regarded as high. The shop superintendent, therefore, was asked for an explanation. Some of the problems which were involved in the construction of this graph will be considered now.

Table 37 shows the two series of weekly totals for the fiscal year 1928–1929. Several dates are starred to indicate the weeks containing holidays. The foremen, clerks, and others performing indirect labor, received fixed weekly wages which were not affected by the holidays. The wages of machine operators, however, were calculated on an hourly basis, so that the presence of a holiday during a week tended to diminish the total of their wages for that week. Thus, the pay rolls in the weeks containing holidays presented a different relationship, and, therefore, they were excluded from the calculations.

In order to simplify the calculations, the decimal point in every pay roll figure was shifted one place to the left which did not change the relationship.

After a critical examination of the dot graph, Fig. 46, it appeared that the calculated straight line was not the most representative expression of the average relationship. In the first place, since both series had extreme items which were out of line with the rest of the points, the direction of the line of relationship was somewhat distorted. Hence, the presence of the extreme items above the middle of the line meant that, when a straight line of relationship was fitted to the data, the line was at a higher level than it should be with reference to the more typical items. Another group of extreme items was found just below the right extremity of the line. Their influence was exerted in pulling down the right end of the line. The left end was correspondingly raised, because the central point, being determined by the averages fo Y-values and of X-values, was relatively little affected by those extreme items.

In the second place, the statistician decided that, because of the economic factors fundamentally inherent in the series, the line of average relationship should not be a straight line. Though the amount of direct labor always fluctuated in proportion to the amount of work in the shop, this was not the case with the fluctuations in the indirect labor. A certain minimum of indirect labor force was maintained in the shop even when the work was temporarily suspended. As the shop activities revived,

and machine operators returned to their jobs, the indirect labor force for a while remained stationary because the same number of men, some of whom had remained idle, were able to perform the increasing amount of indirect labor. The result was that the slope of the line of relationship was practically horizontal. Then, as activity increased, the slope of the line of relationship became steeper. But when the factory began to receive an abnormally large number of orders, increases in the indirect labor force again lagged behind those of the direct labor force so that the slope of the line tended to become less steep. This was because the foremen and clerks were required to do a greater amount of work per man. An abnormally busy time was not expected to last, and the hiring of extra men for indirect labor was avoided. Moreover, the overtime work did not affect the weekly wages of those who received them as fixed amounts irrespective of the fluctuations in the amount of work performed by each man.

In order to determine a more typical line of average relationship, the statistician fitted a curved line (see Fig. 45). Because he fitted it by a free-hand visual method, he unconsciously assigned less weight to those items which he considered not sufficiently typical since they were out of line with the rest.

TABLE 37

Direct and Indirect Labor Pay Rolls

Weekly Totals

Ray Radio Company

Date	Direct Labor, Dollars	Indirect Labor, Dollars	Date	Direct Labor, Dollars	Indirect Labor, Dollars
1928			1929		
July 4*..	1,490	900	January 2*.	1,030	790
11...	1,540	970	9..	1,320	900
18...	1,540	990	16..	1,250	890
25...	1,510	940	23..	1,240	860
			30..	1,280	860
August 1...	1,600	960			
8...	1,500	970	February 6..	1,190	830
15...	1,460	990	13..	1,210	860
22...	1,410	1,000	20..	1,270	880
29...	1,450	1,000	27*.	1,130	760
September 5...	1,390	960	March 6..	1,380	910
12*..	1,170	800	13..	1,410	930
19...	1,390	960	20..	1,410	910
26...	1,390	970	27..	1,490	910
October 3...	1,390	920	April 3..	1,460	920
10...	1,400	960	10..	1,440	930
17*..	1,110	790	17..	1,530	940
24...	1,360	880	24*.	1,370	820
31...	1,320	880			
			May 1..	1,670	1,010
November 7...	1,270	870	8..	1,670	1,020
14...	1,280	850	15..	1,740	1,030
21...	1,310	840	22..	1,840	1,010
28*..	1,070	720	29..	1,900	1,020
December 5...	1,290	880	June 5*.	1,530	870
12...	1,310	880	12..	1,870	1,040
19...	1,420	910	19..	1,750	1,030
26*..	970	660	26..	1,640	1,000

* Weeks containing holidays are starred. These pay rolls are not included in calculations.

TABLE 38

Correlation of Direct and Indirect Labor

Preliminary Calculations

(Zero in Original Data Omitted)

Direct Labor X	Indirect Labor Y	XY	X^2	Y^2	
154	97	14,938	23,716	9,409	
154	99	15,246	23,716	9.801	
151	94	14,194	22,801	8,836	
160	96	15,360	25,600	9,216	
150	97	14,550	22,500	9,409	
146	99	14,454	21,316	9,801	
141	100	14,100	19,881	10,000	
145	100	14,500	21,025	10,000	
139	96	13,344	19,321	9,216	
139	96	13,344	19,321	9,216	
139	97	13,483	19,321	9,409	
139	92	12,788	19,321	8,464	
140	96	13,440	19,600	9,216	
136	88	11,968	18,496	7,744	
132	88	11,616	17,424	7,744	
127	87	11,049	16,129	7,569	
128	85	10,880	16,384	7,225	
131	84	11,004	17,161	7,056	
129	88	11,352	16,641	7,744	
131	88	11,528	17,161	7,744	
142	91	12,922	20,164	8,281	
132	90	11,880	17,424	8,100	
125	89	11,125	15,625	7,921	
124	86	10,664	15,376	7,396	
128	86	11,008	16,384	7,396	
119	83	9,877	14,161	6,889	
121	86	10,406	14,641	7,396	
127	88	11,176	16,129	7,744	
138	91	12,558	19,044	8,281	
141	93	13,113	19,881	8,649	
141	91	12,831	19,881	8,281	
149	91	13,559	22,201	8,281	
146	92	13,432	21,316	8,464	
144	93	13,392	20,736	8,649	
153	94	14.382	23,409	8,836	
167	101	16,867	27,889	10,201	
167	102	17,034	27,889	10,404	
174	103	17,922	30,276	10,609	
184	101	18,584	33,856	10,201	
190	102	19,380	36,100	10,404	
187	104	19,448	34,969	10,816	
175	103	18,025	30,625	10,609	
164	100	16,400	26,896	10,000	
ΣX	ΣY	ΣXY	ΣX^2	ΣY^2	N
6,249	4,027	589,123	921,707	378,627	43
$(\Sigma X)^2$	$(\Sigma Y)^2$				
39,050,001	16,216,729				

TABLE 39

Correlation of Direct and Indirect Labor

Calculation of Coefficient of Correlation

Ayres' formula for coefficient of correlation (r)

$$r = \frac{\Sigma XY - \dfrac{\Sigma X \Sigma Y}{N}}{\sqrt{\Sigma X^2 - \dfrac{(\Sigma X)^2}{N}}\sqrt{\Sigma Y^2 - \dfrac{(\Sigma Y)^2}{N}}}$$

$$r = \frac{589{,}123 - \dfrac{6{,}249}{43} \times 4{,}027}{\sqrt{921{,}707 - \dfrac{39{,}050{,}001}{43}}\sqrt{378{,}627 - \dfrac{16{,}216{,}729}{43}}}$$

$$= \frac{589{,}123 - \dfrac{25{,}164{,}723}{43}}{\sqrt{921{,}707 - 908{,}140}\sqrt{378{,}627 - 377{,}133}}$$

$$= \frac{589{,}123 - 585{,}226}{\sqrt{13{,}567}\sqrt{1{,}494}} = \frac{3{,}897}{\sqrt{(13{,}567)(1{,}494)}}$$

$$= \frac{3{,}897}{\sqrt{20{,}269{,}098}}$$

$$= \frac{3{,}897}{4{,}502}$$

or

$$r = +0.8656$$

TABLE 40

Correlation of Direct and Indirect Labor
Calculation of Line of Relationship*

Ayres' formula: $$y = \frac{\Sigma XY - \frac{\Sigma X \Sigma Y}{N}}{\Sigma X^2 - \frac{(\Sigma X)^2}{N}} x$$

Taking values from the calculation of coefficient of correlation by the Ayres' formula

$$y = +\frac{3{,}897}{13{,}567} = +0.287x$$

if $x = 1$; $y = +0.287$
if $x = 50$; $y = 14.4$

Calculation of the central point:

$$\frac{X}{N} = \frac{6{,}249}{43} = 145.3 = \text{average of } X \text{ values}$$

$$\frac{Y}{N} = \frac{4{,}027}{43} = 93.7 = \text{average of } Y \text{ values}$$

1. † (When $X = 145.3$; $Y = 93.7$)

Calculation of a second point on the line:

(Taking 50 for x, y equals 14.4)

Average $X + x = 145.3 + 50 = 195.3$
Average $Y + y = 93.7 + 14.4 = 108.1$

2. † (When $X = 195.3$; $Y = 108.1$)

Illustration of calculation for Y when X is given. $X = 150$*

$$\frac{\Sigma X}{N} = 145.3$$

$$\frac{\Sigma Y}{N} = 93.7$$

$$Y - \frac{\Sigma Y}{N} = \frac{\Sigma XY - \frac{\Sigma X \Sigma Y}{N}}{\Sigma X^2 - \frac{(\Sigma X)^2}{N}} \cdot \left(X - \frac{\Sigma X}{N}\right)$$

$$Y - 93.7 = 0.287(150 - 145.3)$$
$$Y = 1.3 + 93.7$$
$$Y = 95$$

Calculation of Standard Error‡

$$S_y^2 = \frac{1}{N}\left\{\Sigma Y^2 - \frac{(\Sigma Y)^2}{N} - \frac{\left[\Sigma XY - \frac{\Sigma X \Sigma Y}{N}\right]^2}{\Sigma X^2 - \frac{(\Sigma X)^2}{N}}\right\}$$

$$= \frac{1}{43}\left[1{,}494 - \frac{(3{,}897)^2}{13{,}567}\right]$$

$$= \frac{1}{43}\left[1{,}494 - \frac{15{,}186{,}609}{13{,}567}\right]$$

$$= \frac{1}{43}(1{,}494 - 1{,}119.4) = \frac{374.6}{43} = 8.7$$

$\sqrt{8.7} = 2.95$. Since one zero was omitted, the standard error, $S_y = \$29.50$.

* Zero omitted.
† These two points determine the direction of the line of average relationship.
‡ The numerical equivalents for different sections of this formula were taken from Table 38.

APPENDIX I

CALCULATION SUGGESTIONS

In undertaking any work which involves calculation, statisticians have found that a few simple principles are very effective in increasing the speed and accuracy of their computations. Reference to Fig. 47 will illustrate many of the principles, which will be discussed briefly.

Sackville Water Company
Index of Changes in Cost of Plant
Weighted Average of Price Relatives, 1923
(June 1914 = 100)

	Relative Prices Lead	Relative Prices Pig Iron	Relative Prices Labor	Weighted Relatives Lead wt. 3%	Weighted Relatives Pig Iron wt. 60%	Weighted Relatives Labor wt. 37%	Index
1923							
Jan.	201	206	206	6.0	123.6	76.2	205.8
Feb.	215	210	204	6.4	126.0	75.5	207.9
Mar.	219	232	212	6.6	139.2	78.4	224.2
Apr.	211	238	213	6.3	142.8	78.8	227.9
May	187	229	218	5.6	137.4	80.7	223.7
Jun.	181	218	219	5.4	130.8	81.0	217.2
Jul.	161	200	217	4.8	120.0	80.3	205.1
Aug.	171	194	214	5.1	116.4	79.2	200.7
Sep.	177	193	216	5.3	115.8	80.0	201.1
Oct.	173	180	218	5.2	108.0	80.7	193.9
Nov.	174	165	218	5.2	99.0	80.7	184.9
Dec.	196	169	220	5.9	101.4	81.4	188.7

Calculation of Weights

	Base Price (June, 1914)	times	Quantity in Base Year (1914)	Initial Cost per Mile	Percent of Total Cost per Mile
Lead	\$ .038 per pound	x	1500 pounds	\$ 57.00	3 %
Pig Iron	\$ 12.97 per ton	x	79 tons	\$ 1024.63	60 %
Labor	\$ 12.70 per week	x	50 man weeks	\$ 635.00	37 %
			Total Cost per Mile	\$ 1716.63	100 %

Source of price data: Standard Trade and Securities Service

FIG. 47.

One of the basic necessities for calculation work is neatness. Although inaccuracy in calculation is not inevitably tied up with the neatness of the work, experience shows that neat work is likely to be more accurate than work which is poorly written. Neatness has two elements; one is the organization of the work, and the other is the style of writing. Of these two,

the first is the more important and is within easy control of every individual.

Work should be organized so that at any time during the process of calculation or afterward any details may be traced easily. This means that the arrangement of tables necessarily must be planned in advance. Planning requires attention to the number and arrangement of columns in order to make sure that all necessary spaces will be provided for on each page.

When values of a series for a number of years are listed, it is worth while to break the years after every group of five. In the case of monthly data, it is well to leave additional space between successive blocks of values for each three months as shown in Fig. 47.

The calculations should be arranged so that they need to be recorded only once. Pencil rather than ink should be used so that erasures may be made easily. An experienced computer may use ink which has the advantage of being permanent and less likely to be smudged.

After the tables have been arranged, titles similar to those described for drawings should be put at the top of the table and the source indicated at the bottom. This will enable one to return to the table months after it has been computed and to understand at once what the table signifies.

The figure illustrates not only the general principles indicated above but also the use of a standard type of calculating paper. The advantage of this paper consists of its uniform size and punching so that a number of sheets may be bound together in book form.

The ruling has been spaced so that the paper may be used in a typewriter equipped with pica-sized type. Three typed letters or numbers will just go in a rectangle. In addition, the vertical width of the rectangles is just enough so that the ruled lines conform to the spacing between lines on the typewriter. Similarly, when figures are written by hand it will be found convenient to write three digits in each rectangle. Decimal points should be aligned vertically. Coordination in planning between the typewriter and written requirements enable the computer to work out a computation which then can be turned over to a typist with assurance that, if the spacing used is followed, the typed copy of the completed table will have the same form and arrangement as the written one.

CALCULATION AIDS

In order to save time in performing necessary calculations, various devices have been invented. These may be divided roughly into three classes: graphs, tables, and machines.

When only approximate values are desired, a graph such as a dot graph on arithmetic paper or a frequency distribution on probability paper, may be entirely satisfactory. For quick estimates, values read graphically are particularly useful. Obviously, the accuracy of the estimate will depend upon the scale used. Given a sufficiently large area on which to draw a graph, almost any reasonable degree of accuracy can be obtained in a given case.

Although tables are computed to aid in calculating work, their function is slightly different from graphs. One type of table is used purely as a means of shortening calculation operations. Tables of logarithms illustrate this type. We are not at all interested in the computations which lead to the construction of the table, but only in the use of the logarithms when they shorten the processes of multiplication, division, or the calculation of powers and roots. A second type of table is found which saves time because it presents in printed, immediately usable form the results of calculations which have to be made frequently. A table of this type is a compound interest table or an annuity table or a stock yield table. Here the purpose is to have available the *results* of certain computations, so that this type of table makes available in systematic form a vast amount of computing experience, usually based on some mathematical law.

When calculations are varied so that tabular values are of little assistance, a calculating machine probably will be found more desirable. Such machines may be divided roughly into two groups: those which are designed to secure a maximum economy of effort in addition and subtraction and those which are designed to secure this economy in multiplication and division.

Most machines now on the market are designed for use in connection with accounting problems in business so that the digit capacity of the machines is very large. If, however, the computer is satisfied with values including accuracy to three digits, a slide rule will be found not only fast but accurate to that extent. This device is in common use by engineers. Within the last few years, business men have found it of increasing value in checking figures or in making estimates.

It is obvious that at times a combination of calculation aids may be desirable. Thus, it is entirely possible that in computing values from a formula in some statistical problem, a table of logarithms, a calculating machine, and a slide rule may all be used to secure a desired figure with a smaller expenditure of effort than through the use of any one separately. Each calculation aid should be used for that purpose to which it is best adapted.

APPENDIX II

ORDINATES OF THE NORMAL CURVE

Expressed as Fractional Parts of the Maximum Ordinate

x/σ	y/y_o	x/σ	y/y_o
0.0	1.00000	2.5	0.04394
0.1	0.99501	2.6	0.03405
0.2	0.98020	2.7	0.02612
0.3	0.95600	2.8	0.01984
0.4	0.92312	2.9	0.01492
0.5	0.88250	3.0	0.01111
0.6	0.83527	3.1	0.00819
0.7	0.78270	3.2	0.00598
0.8	0.72615	3.3	0.00432
0.9	0.66689	3.4	0.00309
1.0	0.60653	3.5	0.00219
1.1	0.54607	3.6	0.00153
1.2	0.48675	3.7	0.00106
1.3	0.42956	3.8	0.00073
1.4	0.37531	3.9	0.00050
1.5	0.32465	4.0	0.00034
1.6	0.27804	4.1	0.00022
1.7	0.23575	4.2	0.00015
1.8	0.19790	4.3	0.00010
1.9	0.16448	4.4	0.00006
2.0	0.13534	4.5	0.00004
2.1	0.11025	4.6	0.00003
2.2	0.08892	4.7	0.00002
2.3	0.07100	4.8	0.00001
2.4	0.05614	4.9	0.00001
....	....	5.0	0.00000

The values of x/σ are deviations from the mean expressed in terms of standard deviation, σ.

The values of y/y_o are the corresponding ordinates expressed in terms of the maximum ordinate, y_o.

The maximum ordinate, $y_o = \dfrac{N}{2.5066\sigma}$, where N is the total frequency, and σ = standard deviation in terms of class intervals.

APPENDIX III

TABLE OF AREAS UNDER THE NORMAL CURVE

Showing the area between the maximum ordinate erected at the arithmetic mean and ordinates erected at x/σ.

x/σ	Area	x/σ	Area
0.00	0.00000	0.45	0.17364
0.01	0.00339	0.50	0.19146
0.02	0.00798	0.55	0.20884
0.03	0.01197	0.60	0.22575
0.04	0.01595	0.65	0.24215
0.05	0.01994	0.70	0.25804
0.06	0.02392	0.75	0.27337
0.07	0.02790	0.80	0.28814
0.08	0.03188	0.85	0.30234
0.09	0.03586	0.90	0,31594
0.10	0.03983	0.95	0.32894
0.11	0.04380	1.00	0.34134
0.12	0.04776	1.10	0.36433
0.13	0.05172	1.20	0.38493
0.14	0.05567	1.25	0.39435
0.15	0.05962	1.30	0.40320
0.16	0.06356	1.40	0.41924
0.17	0.06749	1.50	0.43319
0.18	0.07142	1.60	0.44520
0.19	0.07535	1.70	0.45543
0.20	0.07926	1.75	0.45994
0.21	0.08317	0.80	0.46407
0.22	0.08706	1.90	0.47128
0.23	0.09095	2.00	0.47725
0.24	0.09483	2.10	0.48214
0.25	0.09871	2.20	0.48610
0.26	0.10257	2.30	0.48928
0.27	0.10642	2.40	0.49180
0.28	0.11026	2.50	0.49379
0.29	0,11409	2.60	0.49534
0.30	0.11791	2.70	0.49653
0.31	0.12172	2.80	0.49744
0.32	0.12552	2.90	0.49813
0.33	0.12930	3.00	0.49865
0.34	0.13307	3.25	0.49942
0.35	0.13683	3.50	0.49977
0.36	0.14058	3.75	0.49991
0.37	0.14431	4.00	0.49997
0.38	0.14803	4.25	0.49999
0.39	0.15173	4.50	0.50000
0.40	0.15542	5.00	0.50000

APPENDIX IV

USE OF LOGARITHMS

In order to carry through certain calculations such as those described in the chapter on Trend Lines, it is essential to know how to use logarithms. Logarithms are merely a numerical device which simplifies calculation where products or factors are involved. Thus, the product of four factors can be obtained by adding the four corresponding logarithms and finding the answers in the table. One simple operation in addition thus replaces three multiplications. Similar statements may be made in regard to division, to the raising of numbers to a given power, and to the finding roots of numbers.

Basic ideas in regard to logarithms can be understood from two simple statements. These are:

$$10^2 = 100$$
$$\log_{10} 100 = 2$$

The first will be recognized as an accurate statement of a simple arithmetic example. The second is to be regarded as another statement which is equivalent to the first. Often students try to derive this second statement from the first by the use of some process of reasoning. This is not the correct point of view. The second statement, if generalized, is simply a definition of a logarithm. It expresses the same facts as the first in different words. It reads: log of 100 is 2.

From the two statements it will be discovered that a logarithm really is a power to which a number is raised in order to make it equal to another number. We commonly use 10 as the number which is to be raised to a power because our number system is based on 10 digits. It must be understood also that we can have logarithms of positive numbers only. For exact powers of 10, the following table will be obvious:

$10^1 = 10$	$\log 10 = 1$
$10^2 = 100$	$\log 100 = 2$
$10^3 = 1{,}000$	$\log 1{,}000 = 3$
$10^4 = 10{,}000$	$\log 10{,}000 = 4$
$10^5 = 100{,}000$	$\log 100{,}000 = 5$

Numbers intermediate between any two of those given in the table can be expressed by raising 10 to some power which will not be a whole number. Thus, $10^{1.301030} = 20$. This is equivalent to saying that $\log 20 = 1.301030$, which is correct to six decimal places. The process of calculating these intermediate powers need not be considered here, since we are interested only in the use of the table. Actually, these powers are calculated by the use of certain algebraic series.

By the foregoing statements, we have discovered that common logarithms represent exponents of 10. The exponent can be calculated to any number of decimal places that a given accuracy may demand. If we stopped with this statement, we should need a very bulky book to contain the logarithms of all numbers. Let us see how we may cut down our prospective book of logarithms to one which is convenient and practical in size.

Take, for example, the use of logarithms for the purpose of multiplication.

From the table of powers of 10 given above we observe the following:

$$10^2 = 100$$
$$10^3 = 1{,}000$$

Multiplying both sides of these equations together we have

$$10^2 \times 10^3 = 100 \times 1{,}000 = 100{,}000$$

We know that 100,000, expressed as a power of 10, can be written 10^5. We note that 5, which is the exponent of 10, can be found by adding the 2 and the 3 on the left-hand side of the equation. From this we may recall the principle that, if we wish to multiply two numbers which are expressed as the powers of a given number (10), we can do it by adding the exponents. Thus, $10^2 \times 10^3 = 10^{2+3} = 10^5$ or, more generally, $10^x \times 10^y = 10^{x+y}$. In terms of logarithms, the statement is equivalent to saying

$$\log (100 \times 1{,}000) = \log 100 + \log 1{,}000 = 2 + 3 = 5$$

or

$$\log (10^x \times 10^y) = \log 10^x + \log 10^y = x + y$$

This illustrates the fundamental principle that, if we wish to multiply two or more numbers together, all that we need to do is to add their logarithms and then look in a table of logarithms for the answer corresponding to this sum.

By the use of the foregoing principle, if we know the logarithms of all numbers between 1 and 10 carried to as many decimal places as we desire, we can find the logarithms of all positive numbers. Thus, log 20 may be found from log 2 because log 20 equals log $(10 \times 2) = \log 10 + \log 2 = 1 + \log 2 = 1 + .301030$. If we wish to find the logarithm of 3,574, we know that

$$3{,}574 = 10^3 \times 3.574.$$

Hence,

$$\begin{aligned} \log 3{,}574 &= \log 10^3 + \log 3.574 \\ &= 3 + \log 3.574 \\ &= 3.55315 \end{aligned}$$

Similarly

$$\begin{aligned} \log 35{,}740 &= 4 + \log 3.574 \\ &= 4.55315 \\ \log 357{,}400 &= 5 + \log 3.574 \\ &= 5.55315 \end{aligned}$$

The logarithm of any positive number can be obtained simply by adding a certain whole number to the logarithm of the number between 1 and 10 which corresponds to the original number. From this it is evident that the size of our table of logarithms can be reduced to a table of decimal exponents representing all of the numbers between 1 and 10 carried to as many decimal places as we desire.

Two steps, then, are necessary in finding the logarithm which corresponds to a given number. These are:

(1) Find from a table of logarithms the decimal part of the power to which 10 is to be raised. This is called the mantissa of the logarithm.

(2) Write down in front of the mantissa the whole number which represents the next lower power of 10. Thus, if the number is 1,378, the next lower power of 10 is 1,000 and the power is 3. The whole number, therefore, is 3. This is called the characteristic.

Sometimes a rule is used to determine what the characteristic should be. For numbers equal to or greater than 1, but less than 10, the characteristic is 0. For numbers equal to or greater than 10, but less than 100, it is 1, and so on. In other words subtract 1 from the number of digits to the left of the decimal point to obtain the value of the characteristic. Thus, the characteristic of 23,486.5 is 4 since there are 5 digits to the left of the decimal point.

In order to find the number corresponding to a logarithm we reverse the process. First, we neglect the characteristic and look in the logarithm table for the decimal part or mantissa of the logarithm. This will give us the arrangement of the figures. The table in this text will give the first four significant figures. When that is done, a number of figures which is one greater than the characteristic should be pointed off to the left of the decimal point.

For numbers greater than 0 but less than 1 the following table of powers of 10 will indicate the negative values of the characteristic.

$$10^1 = 10 \qquad \log 10 = 1$$
$$10^0 = 1 \qquad \log 1 = 0$$
$$1/10 = 10^{-1} = .1 \qquad \log .1 = -1$$
$$1/100 = 10^{-2} = .01 \qquad \log .01 = -2$$
$$1/1{,}000 = 10^{-3} = .001 \qquad \log .001 = -3$$

In stating the rules for the characteristic we have not indicated how the characteristic for decimals greater than 0 but less than 1 should be determined. Since the logarithm of 1 equals 0, numbers less than 1 should have a logarithm less than 0, that is, a negative logarithm. Thus, although there are no logarithms of negative numbers, there are negative logarithms. This statement should be clearly understood. If we think, for a moment, of a thermometer scale, we shall recognize that the temperature above and below 0 is measured from the 0 line in *two* different directions, the values below 0 being negative. In the case of logarithms this means that, whenever we have factors which lie between 0 and 1 in value, we must add their negative logarithms in order to get a net total. Furthermore, since the table of logarithms is arranged only for numbers greater than 1, we shall have to perform a subtraction in every case in order to determine the true negative amount.

This difficulty may be overcome by a simple device. We can assign the whole negative value to the characteristic. Thus, $\log 0.2 = \log (0.1 \times 2) = \log 0.1 + \log 2$, but $\log 0.1 + \log 2 = -1 + \log 2 = -1 + 0.301030$. This device gives us negative characteristics but positive mantissas. In order to distinguish the negative characteristic, a minus sign sometimes is put over the characteristic. Such minus signs, however, are likely to be overlooked and, consequently, lead to serious errors. They have the further difficulty of leaving us with negative and positive

characteristics which must be combined, thus making another possible source of error. To avoid this the characteristic for 0.2 will be written 9.000 − 10 so that log 0.2 will be written 9.301030 − 10. Similarly, log 0.02 = 8.301030 − 10, and so on. A rule for the characteristics of decimal numbers can be derived. Count the number of zeros to the right of the decimal point up to the first significant number. Increase the number by 1 and subtract from 10. This gives a positive value to the characteristic. Then subtract 10 from the whole logarithm.

In the table it will be observed that the logarithms can be found directly only for numbers that have four digits or less. When the logarithms of numbers of five or more digits are desired, the result has to be obtained by a process of interpolation since the fifth digit represents the pro rata or proportional position between two other logarithms. Thus, log 22,164 lies four-tenths of the way from log 22,160 to log 22,170. From the table it will be found that the first logarithm is 4.34557 and the second is 4.34577. The difference between the logarithms, therefore, is 20. At the side of the table there are supplementary tables of proportional parts which are merely prorata tables. Under the adjacent small table, entitled 20, opposite number 4 will be found 8. This means that, if 8 is added to the last digit of the mantissa of the smaller logarithm, we shall find the logarithm we desire. Thus, the desired logarithm is 4.34565. The reverse process is used when finding a number corresponding to a logarithm. Thus, a logarithm which is 1.37754 corresponds to a number 23.853.

Example 1. Verify

log 478 = 2.67943
log 7.63 = 0.88252
log 24,780 = 4.39410
log 0.0478 = 8.67943 − 10
log 49.68 = 1.69618
log 1003 = 3.00130
log 1.030 = 0.01284

Example 2. Find the numbers corresponding to the following logarithms

7.06070 − 10
2.79414
3.94729
0.47987
1.68543
9.51032 − 10

We have shown how to look up the logarithms corresponding to certain numbers and the reverse process. We have also indicated that numbers may be multiplied by adding their logarithms. The following examples will illustrate the process.

Example 3. Calculate the value of the following product

$$128 \times 315$$
$$\begin{aligned} \text{Log } 128 &= 2.10721 \\ \text{Log } 315 &= 2.49831 \\ \text{Log of product} &= 4.60552 \\ \text{Product} &= 40{,}320 \end{aligned}$$

Example 4. Find the value of the product

$$12{,}478 \times 0.54321$$
$$\begin{aligned} \text{Log } 12{,}478 &= 4.09614 \\ \text{Log } 0.54321 &= 9.73497 - 10 \\ \text{Log of product} &= 3.83111 \\ \text{Product} &= 6778.1 \end{aligned}$$

When we wish to divide, the logarithm of the divisor is subtracted from the logarithm of the dividend and the result is the logarithm of the quotient. The following examples will illustrate the process.

Example 5. Divide 625 by 15

$$\begin{aligned} \text{Log } 625 &= 2.79588 \\ \text{Log } 15 &= 1.17609 \\ \text{Log of quotient} &= 1.61979 \\ \text{Quotient} &= 41.667 \end{aligned}$$

Example 6. Divide .9784 by .0046

$$\begin{aligned} \text{Log } 0.9784 &= 9.99052 - 10 \\ \text{Log } 0.0046 &= 7.66276 - 10 \\ \text{Log of quotient} &= 2.32776 \\ \text{Quotient} &= 212.69 \end{aligned}$$

Since 15^3 equals $15 \times 15 \times 15$ and a similar relation holds for any other power, we can find the logarithm of 15^3 by adding together three of the logarithms of 15. This is equivalent to saying that we multiply the logarithm of 15 by 3. In other words, in order to raise a number to a power we multiply the logarithm of the number by the power. An example will show the process.

Example 7. Find the value of 5^7

$$\begin{aligned} \text{Log } 5^7 &= 7 \log 5 \\ &= 7\ (0.69897) \\ &= 4.89279 \\ 5^7 &= 78{,}125 \end{aligned}$$

Example 8. Find the value of $16^5 \times 3^4$

$$\begin{aligned}\text{Log of product} &= 5 \log 16 + 4 \log 3 \\ &= 5\ (1.20412) + 4\ (0.47712) \\ &= 6.02060 + 1.90848 \\ &= 7.92908 \\ \text{Product} &= 84{,}934{,}000\end{aligned}$$

Since the calculation of a root, such as the square root of a number, is the inverse of the process found in raising a number to a power, we find the square root of a number by dividing the logarithm by 2. In some calculations, such as the link relative method of calculating seasonal, we have to take the twelfth root of a number because there are 12 months in the year. This is done by dividing the logarithm by 12. Examples will show the process.

Example 9. Find the value of $\sqrt[7]{128}$

$$\begin{aligned}\text{Log} \sqrt[7]{128} &= \tfrac{1}{7} \log 128 \\ &= \tfrac{1}{7}\ (2.10721) \\ &= 0.30103 \\ \sqrt[7]{128} &= 2\end{aligned}$$

Example 10. Find the value of $\sqrt[3]{13.748}$

$$\begin{aligned}\text{Log} \sqrt[3]{13.748} &= \tfrac{1}{3} \log 13.748 \\ &= \tfrac{1}{3}\ (1.13824) = 0.37941 \\ \sqrt[3]{13.748} &= 2.3956\end{aligned}$$

APPENDIX V

COMMON LOGARITHMS OF NUMBERS[1]

From 1 to 10,000 to five places.
From 10,000 to 11,000 to seven places.

NOTE.—In the tables

* indicates that the first two figures of the mantissa should be taken from the next lower line.

$\bar{5}$ indicates that the mantissa has been rounded off *up* to a 5. Thus

	Seven places		This table
log 1363	3.1344959	=	$3.1345\bar{0}$
log 1402	3.1467480	=	$3.1467\bar{5}$

[1] From Palmer and Leigh, *Plane and Spherical Trigonometry*, McGraw-Hill Book Company, Inc., New York.

100–150

N.	L. 0	1	2	3	4	5	6	7	8	9
100	00 000	043	087	130	173	217	260	303	346	389
101	432	475	518	561	604	647	689	732	775	817
102	860	903	945	988	*030	*072	*115	*157	*199	*242
103	01 284	326	368	410	452	494	536	578	620	662
104	703	745	787	828	870	912	953	995	*036	*078
105	02 119	160	202	243	284	325	366	407	449	490
106	531	572	612	653	694	735	776	816	857	898
107	938	979	*019	*060	*100	*141	*181	*222	*262	*302
108	03 342	383	423	463	503	543	583	623	663	703
109	743	782	822	862	902	941	981	*021	*060	*100
110	04 139	179	218	258	297	336	376	415	454	493
111	532	571	610	650	689	727	766	805	844	883
112	922	961	999	*038	*077	*115	*154	*192	*231	*269
113	05 308	346	385	423	461	500	538	576	614	652
114	690	729	767	805	843	881	918	956	994	*032
115	06 070	108	145	183	221	258	296	333	371	408
116	446	483	521	558	595	633	670	707	744	781
117	819	856	893	930	967	*004	*041	*078	*115	*151
118	07 188	225	262	298	335	372	408	445	482	518
119	555	591	628	664	700	737	773	809	846	882
120	918	954	990	*027	*063	*099	*135	*171	*207	*243
121	08 279	314	350	386	422	458	493	529	565	600
122	636	672	707	743	778	814	849	884	920	955
123	991	*026	*061	*096	*132	*167	*202	*237	*272	*307
124	09 342	377	412	447	482	517	552	587	621	656
125	691	726	760	795	830	864	899	934	968	*003
126	10 037	072	106	140	175	209	243	278	312	346
127	380	415	449	483	517	551	585	619	653	687
128	721	755	789	823	857	890	924	958	992	*025
129	11 059	093	126	160	193	227	261	294	327	361
130	394	428	461	494	528	561	594	628	661	694
131	727	760	793	826	860	893	926	959	992	*024
132	12 057	090	123	156	189	222	254	287	320	352
133	385	418	450	483	516	548	581	613	646	678
134	710	743	775	808	840	872	905	937	969	*001
135	13 033	066	098	130	162	194	226	258	290	322
136	354	386	418	450	481	513	545	577	609	640
137	672	704	735	767	799	830	862	893	925	956
138	988	*019	*051	*082	*114	*145	*176	*208	*239	*270
139	14 301	333	364	395	426	457	489	520	551	582
140	613	644	675	706	737	768	799	829	860	891
141	922	953	983	*014	*045	*076	*106	*137	*168	*198
142	15 229	259	290	320	351	381	412	442	473	503
143	534	564	594	625	655	685	715	746	776	806
144	836	866	897	927	957	987	*017	*047	*077	*107
145	16 137	167	197	227	256	286	316	346	376	406
146	435	465	495	524	554	584	613	643	673	702
147	732	761	791	820	850	879	909	938	967	997
148	17 026	056	085	114	143	173	202	231	260	289
149	319	348	377	406	435	464	493	522	551	580
150	609	638	667	696	725	754	782	811	840	869
N.	L. 0	1	2	3	4	5	6	7	8	9

Prop. Parts

	44	43	42
1	4.4	4.3	4.2
2	8.8	8.6	8.4
3	13.2	12.9	12.6
4	17.6	17.2	16.8
5	22.0	21.5	21.0
6	26.4	25.8	25.2
7	30.8	30.1	29.4
8	35.2	34.4	33.6
9	39.6	38.7	37.8

	41	40	39
1	4.1	4.0	3.9
2	8.2	8.0	7.8
3	12.3	12.0	11.7
4	16.4	16.0	15.6
5	20.5	20.0	19.5
6	24.6	24.0	23.4
7	28.7	28.0	27.3
8	32.8	32.0	31.2
9	36.9	36.0	35.1

	38	37	36
1	3.8	3.7	3.6
2	7.6	7.4	7.2
3	11.4	11.1	10.8
4	15.2	14.8	14.4
5	19.0	18.5	18.0
6	22.8	22.2	21.6
7	26.6	25.9	25.2
8	30.4	29.6	28.8
9	34.2	33.3	32.4

	35	34	33
1	3.5	3.4	3.3
2	7.0	6.8	6.6
3	10.5	10.2	9.9
4	14.0	13.6	13.2
5	17.5	17.0	16.5
6	21.0	20.4	19.8
7	24.5	23.8	23.1
8	28.0	27.2	26.4
9	31.5	30.6	29.7

	32	31	30
1	3.2	3.1	3.0
2	6.4	6.2	6.0
3	9.6	9.3	9.0
4	12.8	12.4	12.0
5	16.0	15.5	15.0
6	19.2	18.6	18.0
7	22.4	21.7	21.0
8	25.6	24.8	24.0
9	28.8	27.9	27.0

Prop. Parts

150–200

N.	L. 0	1	2	3	4	5	6	7	8	9
150	17 609	638	667	696	72$\bar{5}$	754	782	811	840	869
151	898	926	955	984	*013	*041	*070	*099	*127	*156
152	18 184	213	241	270	298	327	355	384	412	441
153	469	498	526	554	583	611	639	667	696	724
154	752	780	808	837	86$\bar{5}$	893	921	949	977	*005
155	19 033	061	089	117	145	173	201	229	257	28$\bar{5}$
156	312	340	368	396	424	451	479	507	53$\bar{5}$	562
157	590	618	645	673	700	728	756	783	811	838
158	866	893	921	948	976	*003	*030	*058	*085	*112
159	20 140	167	194	222	249	276	303	330	358	38$\bar{5}$
160	412	439	466	493	520	548	57$\bar{5}$	602	629	656
161	683	710	737	763	790	817	844	871	898	92$\bar{5}$
162	952	978	*005	*032	*059	*085	*112	*139	*165	*192
163	21 219	245	272	299	325	352	378	40$\bar{5}$	431	458
164	484	511	537	564	590	617	643	669	696	722
165	748	77$\bar{5}$	801	827	854	880	906	932	958	98$\bar{5}$
166	22 011	037	063	089	115	141	167	194	220	246
167	272	298	324	3$\bar{5}$0	376	401	427	453	479	505
168	531	557	583	608	634	660	686	712	737	763
169	789	814	840	866	891	917	943	968	994	*019
170	23 04$\bar{5}$	070	096	121	147	172	198	223	249	274
171	300	325	350	376	401	426	452	477	502	528
172	553	578	603	629	654	679	704	729	754	779
173	80$\bar{5}$	830	85$\bar{5}$	880	90$\bar{5}$	930	95$\bar{5}$	980	*00$\bar{5}$	*030
174	24 05$\bar{5}$	080	10$\bar{5}$	130	15$\bar{5}$	180	204	229	254	279
175	304	329	353	378	403	428	452	477	502	527
176	551	576	601	625	6$\bar{5}$0	674	699	724	748	773
177	797	822	846	871	895	920	944	969	993	*018
178	25 042	066	091	115	139	164	188	212	237	261
179	285	310	334	358	382	406	431	45$\bar{5}$	479	503
180	527	551	575	600	624	648	672	696	720	744
181	768	792	816	840	864	888	912	935	959	983
182	26 007	031	05$\bar{5}$	079	102	126	150	174	198	221
183	245	269	293	316	340	364	387	411	43$\bar{5}$	458
184	482	505	529	553	576	600	623	647	670	694
185	717	741	764	788	811	834	858	881	90$\bar{5}$	928
186	951	97$\bar{5}$	998	*021	*04$\bar{5}$	*068	*091	*114	*138	*161
187	27 184	207	231	254	277	300	323	346	370	393
188	416	439	462	485	508	531	554	577	600	623
189	646	669	692	715	738	761	784	807	830	852
190	875	898	921	944	967	989	*012	*035	*058	*081
191	28 103	126	149	171	194	217	240	262	28$\bar{5}$	307
192	330	353	375	398	421	443	466	488	511	533
193	556	578	601	623	646	668	691	713	735	758
194	780	803	82$\bar{5}$	847	870	892	914	937	959	981
195	29 003	026	048	070	092	11$\bar{5}$	137	159	181	203
196	226	248	270	292	314	336	358	380	403	42$\bar{5}$
197	447	469	491	513	53$\bar{5}$	557	579	601	623	64$\bar{5}$
198	667	688	710	732	754	776	798	820	842	863
199	885	907	929	951	973	994	*016	*038	*060	*081
200	30 103	12$\bar{5}$	146	168	190	211	233	25$\bar{5}$	276	298
N.	L. 0	1	2	3	4	5	6	7	8	9

Prop. Parts

	29	28
1	2.9	2.8
2	5.8	5.6
3	8.7	8.4
4	11.6	11.2
5	14.5	14.0
6	17.4	16.8
7	20.3	19.6
8	23.2	22.4
9	26.1	25.2

	27	26
1	2.7	2.6
2	5.4	5.2
3	8.1	7.8
4	10.8	10.4
5	13.5	13.0
6	16.2	15.6
7	18.9	18.2
8	21.6	20.8
9	24.3	23.4

	25
1	2.5
2	5.0
3	7.5
4	10.0
5	12.5
6	15.0
7	17.5
8	20.0
9	22.5

	24	23
1	2.4	2.3
2	4.8	4.6
3	7.2	6.9
4	9.6	9.2
5	12.0	11.5
6	14.4	13.8
7	16.8	16.1
8	19.2	18.4
9	21.6	20.7

	22	21
1	2.2	2.1
2	4.4	4.2
3	6.6	6.3
4	8.8	8.4
5	11.0	10.5
6	13.2	12.6
7	15.4	14.7
8	17.6	16.8
9	19.8	18.9

Prop. Parts

200–250

N.	L.	0	1	2	3	4	5	6	7	8	9
200	30	103	125	146	168	190	211	233	255	276	298
201		320	341	363	384	406	428	449	471	492	514
202		535	557	578	600	621	643	664	685	707	728
203		750	771	792	814	835	856	878	899	920	942
204		963	984	*006	*027	*048	*069	*091	*112	*133	*154
205	31	175	197	218	239	260	281	302	323	345	366
206		387	408	429	450	471	492	513	534	555	576
207		597	618	639	660	681	702	723	744	765	785
208		806	827	848	869	890	911	931	952	973	994
209	32	015	035	056	077	098	118	139	160	181	201
210		222	243	263	284	305	325	346	366	387	408
211		428	449	469	490	510	531	552	572	593	613
212		634	654	675	695	715	736	756	777	797	818
213		838	858	879	899	919	940	960	980	*001	*021
214	33	041	062	082	102	122	143	163	183	203	224
215		244	264	284	304	325	345	365	385	405	425
216		445	465	486	506	526	546	566	586	606	626
217		646	666	686	706	726	746	766	786	806	826
218		846	866	885	905	925	945	965	985	*005	*025
219	34	044	064	084	104	124	143	163	183	203	223
220		242	262	282	301	321	341	361	380	400	420
221		439	459	479	498	518	537	557	577	596	616
222		635	655	674	694	713	733	753	772	792	811
223		830	850	869	889	908	928	947	967	986	*005
224	35	025	044	064	083	102	122	141	160	180	199
225		218	238	257	276	295	315	334	353	372	392
226		411	430	449	468	488	507	526	545	564	583
227		603	622	641	660	679	698	717	736	755	774
228		793	813	832	851	870	889	908	927	946	965
229		984	*003	*021	*040	*059	*078	*097	*116	*135	*154
230	36	173	192	211	229	248	267	286	305	324	342
231		361	380	399	418	436	455	474	493	511	530
232		549	568	586	605	624	642	661	680	698	717
233		736	754	773	791	810	829	847	866	884	903
234		922	940	959	977	996	*014	*033	*051	*070	*088
235	37	107	125	144	162	181	199	218	236	254	273
236		291	310	328	346	365	383	401	420	438	457
237		475	493	511	530	548	566	585	603	621	639
238		658	676	694	712	731	749	767	785	803	822
239		840	858	876	894	912	931	949	967	985	*003
240	38	021	039	057	075	093	112	130	148	166	184
241		202	220	238	256	274	292	310	328	346	364
242		382	399	417	435	453	471	489	507	525	543
243		561	578	596	614	632	650	668	686	703	721
244		739	757	775	792	810	828	846	863	881	899
245		917	934	952	970	987	*005	*023	*041	*058	*076
246	39	094	111	129	146	164	182	199	217	235	252
247		270	287	305	322	340	358	375	393	410	428
248		445	463	480	498	515	533	550	568	585	602
249		620	637	655	672	690	707	724	742	759	777
250		794	811	829	846	863	881	898	915	933	950
N.	L. 0		1	2	3	4	5	6	7	8	9

Prop. Parts

	22	21
1	2.2	2.1
2	4.4	4.2
3	6.6	6.3
4	8.8	8.4
5	11.0	10.5
6	13.2	12.6
7	15.4	14.7
8	17.6	16.8
9	19.8	18.9

	20
1	2.0
2	4.0
3	6.0
4	8.0
5	10.0
6	12.0
7	14.0
8	16.0
9	18.0

	19
1	1.9
2	3.8
3	5.7
4	7.6
5	9.5
6	11.4
7	13.3
8	15.2
9	17.1

	18
1	1.8
2	3.6
3	5.4
4	7.2
5	9.0
6	10.8
7	12.6
8	14.4
9	16.2

	17
1	1.7
2	3.4
3	5.1
4	6.8
5	8.5
6	10.2
7	11.9
8	13.6
9	15.3

250–300

N.	L. 0	1	2	3	4	5	6	7	8	9	Prop. Parts
250	39 794	811	829	846	863	881	898	915	933	950	**18**
251	967	985̄	*002	*019	*037	*054	*071	*088	*106	*123	1 1.8
252	40 140	157	175̄	192	209	226	243	261	278	295̄	2 3.6
253	312	329	346	364	381	398	415̄	432	449	466	3 5.4
254	483	500	518	535̄	552	569	586	603	620	637	4 7.2
255	654	671	688	705	722	739	756	773	790	807	5 9.0
256	824	841	858	875̄	892	909	926	943	960	976	6 10.8
257	993	*010	*027	*044	*061	*078	*095̄	*111	*128	*145	7 12.6
258	41 162	179	196	212	229	246	263	280	296	313	8 14.4
259	330	347	363	380	397	414	430	447	464	481	9 16.2
260	497	514	531	547	564	581	597	614	631	647	**17**
261	664	681	697	714	731	747	764	780	797	814	1 1.7
262	830	847	863	880	896	913	929	946	963	979	2 3.4
263	996	*012	*029	*045	*062	*078	*095̄	*111	*127	*144	3 5.1
264	42 160	177	193	210	226	243	259	275	292	308	4 6.8
265	325̄	341	357	374	390	406	423	439	455	472	5 8.5
266	488	504	521	537	553	570	586	602	619	635̄	6 10.2
267	651	667	684	700	716	732	749	765̄	781	797	7 11.9
268	813	830	846	862	878	894	911	927	943	959	8 13.6
269	975	991	*008	*024	*040	*056	*072	*088	*104	*120	9 15.3
270	43 136	152	169	185̄	201	217	233	249	265̄	281	log e = 0.43429
271	297	313	329	345̄	361	377	393	409	425̄	441	**16**
272	457	473	489	505̄	521	537	553	569	584	600	1 1.6
273	616	632	648	664	680	696	712	727	743	759	2 3.2
274	775	791	807	823	838	854	870	886	902	917	3 4.8
275	933	949	965̄	981	996	*012	*028	*044	*059	*075	4 6.4
276	44 091	107	122	138	154	170	185	201	217	232	5 8.0
277	248	264	279	295̄	311	326	342	358	373	389	6 9.6
278	404	420	436	451	467	483	498	514	529	545̄	7 11.2
279	560	576	592	607	623	638	654	669	685̄	700	8 12.8
280	716	731	747	762	778	793	809	824	840	855	9 14.4
281	871	886	902	917	932	948	963	979	994	*010	**15**
282	45 025̄	040	056	071	086	102	117	133	148	163	1 1.5
283	179	194	209	225̄	240	255	271	286	301	317	2 3.0
284	332	347	362	378	393	408	423	439	454	469	3 4.5
285	484	500	515̄	530	545	561	576	591	606	621	4 6.0
286	637	652	667	682	697	712	728	743	758	773	5 7.5
287	788	803	818	834	849	864	879	894	909	924	6 9.0
288	939	954	969	984	*000	*015̄	*030	*045̄	*060	*075̄	7 10.5
289	46 090	105̄	120	135̄	150	165̄	180	195̄	210	225̄	8 12.0
290	240	255̄	270	285̄	300	315̄	330	345̄	359	374	9 13.5
291	389	404	419	434	449	464	479	494	509	523	**14**
292	538	553	568	583	598	613	627	642	657	672	1 1.4
293	687	702	716	731	746	761	776	790	805	820	2 2.8
294	835̄	850	864	879	894	909	923	938	953	967	3 4.2
295	982	997	*012	*026	*041	*056	*070	*085	*100	*114	4 5.6
296	47 129	144	159	173	188	202	217	232	246	261	5 7.0
297	276	290	305̄	319	334	349	363	378	392	407	6 8.4
298	422	436	451	465	480	494	509	524	538	553	7 9.8
299	567	582	596	611	625	640	654	669	683	698	8 11.2
300	712	727	741	756	770	784	799	813	828	842	9 12.6
N.	L. 0	1	2	3	4	5	6	7	8	9	Prop. Parts

300–350

N.	L.	0	1	2	3	4	5	6	7	8	9
300	47	712	727	741	756	770	784	799	813	828	842
301		857	871	885	900	914	929	943	958	972	986
302	48	001	015	029	044	058	073	087	101	116	130
303		144	159	173	187	202	216	230	244	259	273
304		287	302	316	330	344	359	373	387	401	416
305		430	444	458	473	487	501	515	530	544	558
306		572	586	601	615	629	643	657	671	686	700
307		714	728	742	756	770	785	799	813	827	841
308		855	869	883	897	911	926	940	954	968	982
309		996	*010	*024	*038	*052	*066	*080	*094	*108	*122
310	49	136	150	164	178	192	206	220	234	248	262
311		276	290	304	318	332	346	360	374	388	402
312		415	429	443	457	471	485	499	513	527	541
313		554	568	582	596	610	624	638	651	665	679
314		693	707	721	734	748	762	776	790	803	817
315		831	845	859	872	886	900	914	927	941	955
316		969	982	996	*010	*024	*037	*051	*065	*079	*092
317	50	106	120	133	147	161	174	188	202	215	229
318		243	256	270	284	297	311	325	338	352	365
319		379	393	406	420	433	447	461	474	488	501
320		515	529	542	556	569	583	596	610	623	637
321		651	664	678	691	705	718	732	745	759	772
322		786	799	813	826	840	853	866	880	893	907
323		920	934	947	961	974	987	*001	*014	*028	*041
324	51	055	068	081	095	108	121	135	148	162	175
325		188	202	215	228	242	255	268	282	295	308
326		322	335	348	362	375	388	402	415	428	441
327		455	468	481	495	508	521	534	548	561	574
328		587	601	614	627	640	654	667	680	693	706
329		720	733	746	759	772	786	799	812	825	838
330		851	865	878	891	904	917	930	943	957	970
331		983	996	*009	*022	*035	*048	*061	*075	*088	*101
332	52	114	127	140	153	166	179	192	205	218	231
333		244	257	270	284	297	310	323	336	349	362
334		375	388	401	414	427	440	453	466	479	492
335		504	517	530	543	556	569	582	595	608	621
336		634	647	660	673	686	699	711	724	737	750
337		763	776	789	802	815	827	840	853	866	879
338		892	905	917	930	943	956	969	982	994	*007
339	53	020	033	046	058	071	084	097	110	122	135
340		148	161	173	186	199	212	224	237	250	263
341		275	288	301	314	326	339	352	364	377	390
342		403	415	428	441	453	466	479	491	504	517
343		529	542	555	567	580	593	605	618	631	643
344		656	668	681	694	706	719	732	744	757	769
345		782	794	807	820	832	845	857	870	882	895
346		908	920	933	945	958	970	983	995	*008	*020
347	54	033	045	058	070	083	095	108	120	133	145
348		158	170	183	195	208	220	233	245	258	270
349		283	295	307	320	332	345	357	370	382	394
350		407	419	432	444	456	469	481	494	506	518
N.	L.	0	1	2	3	4	5	6	7	8	9

Prop. Parts

	15
1	1.5
2	3.0
3	4.5
4	6.0
5	7.5
6	9.0
7	10.5
8	12.0
9	13.5

$\log \pi = 0.49715$

	14
1	1.4
2	2.8
3	4.2
4	5.6
5	7.0
6	8.4
7	9.8
8	11.2
9	12.6

	13
1	1.3
2	2.6
3	3.9
4	5.2
5	6.5
6	7.8
7	9.1
8	10.4
9	11.7

	12
1	1.2
2	2.4
3	3.6
4	4.8
5	6.0
6	7.2
7	8.4
8	9.6
9	10.8

Prop. Parts

350–400

N.	L. 0	1	2	3	4	5	6	7	8	9
350	54 407	419	432	444	456	469	481	494	506	518
351	531	543	555	568	580	593	605	617	630	642
352	654	667	679	691	704	716	728	741	753	765
353	777	790	802	814	827	839	851	864	876	888
354	900	913	925	937	949	962	974	986	998	*011
355	55 023	035	047	060	072	084	096	108	121	133
356	145	157	169	182	194	206	218	230	242	255
357	267	279	291	303	315	328	340	352	364	376
358	388	400	413	425	437	449	461	473	485	497
359	509	522	534	546	558	570	582	594	606	618
360	630	642	654	666	678	691	703	715	727	739
361	751	763	775	787	799	811	823	835	847	859
362	871	883	895	907	919	931	943	955	967	979
363	991	*003	*015	*027	*038	*050	*062	*074	*086	*098
364	56 110	122	134	146	158	170	182	194	205	217
365	229	241	253	265	277	289	301	312	324	336
366	348	360	372	384	396	407	419	431	443	455
367	467	478	490	502	514	526	538	549	561	573
368	585	597	608	620	632	644	656	667	679	691
369	703	714	726	738	750	761	773	785	797	808
370	820	832	844	855	867	879	891	902	914	926
371	937	949	961	972	984	996	*008	*019	*031	*043
372	57 054	066	078	089	101	113	124	136	148	159
373	171	183	194	206	217	229	241	252	264	276
374	287	299	310	322	334	345	357	368	380	392
375	403	415	426	438	449	461	473	484	496	507
376	519	530	542	553	565	576	588	600	611	623
377	634	646	657	669	680	692	703	715	726	738
378	749	761	772	784	795	807	818	830	841	852
379	864	875	887	898	910	921	933	944	955	967
380	978	990	*001	*013	*024	*035	*047	*058	*070	*081
381	58 092	104	115	127	138	149	161	172	184	195
382	206	218	229	240	252	263	274	286	297	309
383	320	331	343	354	365	377	388	399	410	422
384	433	444	456	467	478	490	501	512	524	535
385	546	557	569	580	591	602	614	625	636	647
386	659	670	681	692	704	715	726	737	749	760
387	771	782	794	805	816	827	838	850	861	872
388	883	894	906	917	928	939	950	961	973	984
389	995	*006	*017	*028	*040	*051	*062	*073	*084	*095
390	59 106	118	129	140	151	162	173	184	195	207
391	218	229	240	251	262	273	284	295	306	318
392	329	340	351	362	373	384	395	406	417	428
393	439	450	461	472	483	494	506	517	528	539
394	550	561	572	583	594	605	616	627	638	649
395	660	671	682	693	704	715	726	737	748	759
396	770	780	791	802	813	824	835	846	857	868
397	879	890	901	912	923	934	945	956	966	977
398	988	999	*010	*021	*032	*043	*054	*065	*076	*086
399	60 097	108	119	130	141	152	163	173	184	195
400	206	217	228	239	249	260	271	282	293	304
N.	L. 0	1	2	3	4	5	6	7	8	9

Prop. Parts

	13	12	11	10
1	1.3	1.2	1.1	1.0
2	2.6	2.4	2.2	2.0
3	3.9	3.6	3.3	3.0
4	5.2	4.8	4.4	4.0
5	6.5	6.0	5.5	5.0
6	7.8	7.2	6.6	6.0
7	9.1	8.4	7.7	7.0
8	10.4	9.6	8.8	8.0
9	11.7	10.8	9.9	9.0

400–450

N.	L. 0	1	2	3	4	5	6	7	8	9
400	60 206	217	228	239	249	260	271	282	293	304
401	314	325	336	347	358	369	379	390	401	412
402	423	433	444	455	466	477	487	498	509	520
403	531	541	552	563	574	584	595	606	617	627
404	638	649	660	670	681	692	703	713	724	735
405	746	756	767	778	788	799	810	821	831	842
406	853	863	874	885	895	906	917	927	938	949
407	959	970	981	991	*002	*013	*023	*034	*045	*055
408	61 066	077	087	098	109	119	130	140	151	162
409	172	183	194	204	215	225	236	247	257	268
410	278	289	300	310	321	331	342	352	363	374
411	384	395	405	416	426	437	448	458	469	479
412	490	500	511	521	532	542	553	563	574	584
413	595	606	616	627	637	648	658	669	679	690
414	700	711	721	731	742	752	763	773	784	794
415	805	815	826	836	847	857	868	878	888	899
416	909	920	930	941	951	962	972	982	993	*003
417	62 014	024	034	045	055	066	076	086	097	107
418	118	128	138	149	159	170	180	190	201	211
419	221	232	242	252	263	273	284	294	304	315
420	325	335	346	356	366	377	387	397	408	418
421	428	439	449	459	469	480	490	500	511	521
422	531	542	552	562	572	583	593	603	613	624
423	634	644	655	665	675	685	696	706	716	726
424	737	747	757	767	778	788	798	808	818	829
425	839	849	859	870	880	890	900	910	921	931
426	941	951	961	972	982	992	*002	*012	*022	*033
427	63 043	053	063	073	083	094	104	114	124	134
428	144	155	165	175	185	195	205	215	225	236
429	246	256	266	276	286	296	306	317	327	337
430	347	357	367	377	387	397	407	417	428	438
431	448	458	468	478	488	498	508	518	528	538
432	548	558	568	579	589	599	609	619	629	639
433	649	659	669	679	689	699	709	719	729	739
434	749	759	769	779	789	799	809	819	829	839
435	849	859	869	879	889	899	909	919	929	939
436	949	959	969	979	988	998	*008	*018	*028	*038
437	64 048	058	068	078	088	098	108	118	128	137
438	147	157	167	177	187	197	207	217	227	237
439	246	256	266	276	286	296	306	316	326	335
440	345	355	365	375	385	395	404	414	424	434
441	444	454	464	473	483	493	503	513	523	532
442	542	552	562	572	582	591	601	611	621	631
443	640	650	660	670	680	689	699	709	719	729
444	738	748	758	768	777	787	797	807	816	826
445	836	846	856	865	875	885	895	904	914	924
446	933	943	953	963	972	982	992	*002	*011	*021
447	65 031	040	050	060	070	079	089	099	108	118
448	128	137	147	157	167	176	186	196	205	215
449	225	234	244	254	263	273	283	292	302	312
450	321	331	341	350	360	369	379	389	398	408
N.	L. 0	1	2	3	4	5	6	7	8	9

Prop. Parts

	11	10	9
1	1.1	1.0	0.9
2	2.2	2.0	1.8
3	3.3	3.0	2.7
4	4.4	4.0	3.6
5	5.5	5.0	4.5
6	6.6	6.0	5.4
7	7.7	7.0	6.3
8	8.8	8.0	7.2
9	9.9	9.0	8.1

450–500

N.	L.	0	1	2	3	4	5	6	7	8	9	Prop. Parts
450	65	321	331	341	350	360	369	379	389	398	408	
451		418	427	437	447	456	466	475	485	495	504	
452		514	523	533	543	552	562	571	581	591	600	
453		610	619	629	639	648	658	667	677	686	696	
454		706	715	725	734	744	753	763	772	782	792	
455		801	811	820	830	839	849	858	868	877	887	
456		896	906	916	925	935	944	954	963	973	982	**10**
457		992	*001	*011	*020	*030	*039	*049	*058	*068	*077	1 | 1.0
458	66	087	096	106	115	124	134	143	153	162	172	2 | 2.0
459		181	191	200	210	219	229	238	247	257	266	3 | 3.0
460		276	285	295	304	314	323	332	342	351	361	4 | 4.0
461		370	380	389	398	408	417	427	436	445	455	5 | 5.0
462		464	474	483	492	502	511	521	530	539	549	6 | 6.0
463		558	567	577	586	596	605	614	624	633	642	7 | 7.0
464		652	661	671	680	689	699	708	717	727	736	8 | 8.0
465		745	755	764	773	783	792	801	811	820	829	9 | 9.0
466		839	848	857	867	876	885	894	904	913	922	
467		932	941	950	960	969	978	987	997	*006	*015	
468	67	025	034	043	052	062	071	080	089	099	108	
469		117	127	136	145	154	164	173	182	191	201	
470		210	219	228	237	247	256	265	274	284	293	
471		302	311	321	330	339	348	357	367	376	385	**9**
472		394	403	413	422	431	440	449	459	468	477	1 | 0.9
473		486	495	504	514	523	532	541	550	560	569	2 | 1.8
474		578	587	596	605	614	624	633	642	651	660	3 | 2.7
475		669	679	688	697	706	715	724	733	742	752	4 | 3.6
476		761	770	779	788	797	806	815	825	834	843	5 | 4.5
477		852	861	870	879	888	897	906	916	925	934	6 | 5.4
478		943	952	961	970	979	988	997	*006	*015	*024	7 | 6.3
479	68	034	043	052	061	070	079	088	097	106	115	8 | 7.2
480		124	133	142	151	160	169	178	187	196	205	9 | 8.1
481		215	224	233	242	251	260	269	278	287	296	
482		305	314	323	332	341	350	359	368	377	386	
483		395	404	413	422	431	440	449	458	467	476	
484		485	494	502	511	520	529	538	547	556	565	
485		574	583	592	601	610	619	628	637	646	655	
486		664	673	681	690	699	708	717	726	735	744	**8**
487		753	762	771	780	789	797	806	815	824	833	1 | 0.8
488		842	851	860	869	878	886	895	904	913	922	2 | 1.6
489		931	940	949	958	966	975	984	993	*002	*011	3 | 2.4
490	69	020	028	037	046	055	064	073	082	090	099	4 | 3.2
491		108	117	126	135	144	152	161	170	179	188	5 | 4.0
492		197	205	214	223	232	241	249	258	267	276	6 | 4.8
493		285	294	302	311	320	329	338	346	355	364	7 | 5.6
494		373	381	390	399	408	417	425	434	443	452	8 | 6.4
495		461	469	478	487	496	504	513	522	531	539	9 | 7.2
496		548	557	566	574	583	592	601	609	618	627	
497		636	644	653	662	671	679	688	697	705	714	
498		723	732	740	749	758	767	775	784	793	801	
499		810	819	827	836	845	854	862	871	880	888	
500		897	906	914	923	932	940	949	958	966	975	
N.	L.	0	1	2	3	4	5	6	7	8	9	Prop. Parts

500–550

N.	L.	0	1	2	3	4	5	6	7	8	9	Prop. Parts
500	69	897	906	914	923	932	940	949	958	966	975	
501		984	992	*001	*010	*018	*027	*036	*044	*053	*062	
502	70	070	079	088	096	105	114	122	131	140	148	
503		157	165	174	183	191	200	209	217	226	234	
504		243	252	260	269	278	286	295	303	312	321	
505		329	338	346	355	364	372	381	389	398	406	9
506		415	424	432	441	449	458	467	475	484	492	1 | 0.9
507		501	509	518	526	535	544	552	561	569	578	2 | 1.8
508		586	595	603	612	621	629	638	646	655	663	3 | 2.7
509		672	680	689	697	706	714	723	731	740	749	4 | 3.6
510		757	766	774	783	791	800	808	817	825	834	5 | 4.5
511		842	851	859	868	876	885	893	902	910	919	6 | 5.4
512		927	935	944	952	961	969	978	986	995	*003	7 | 6.3
513	71	012	020	029	037	046	054	063	071	079	088	8 | 7.2
514		096	105	113	122	130	139	147	155	164	172	9 | 8.1
515		181	189	198	206	214	223	231	240	248	257	
516		265	273	282	290	299	307	315	324	332	341	
517		349	357	366	374	383	391	399	408	416	425	
518		433	441	450	458	466	475	483	492	500	508	
519		517	525	533	542	550	559	567	575	584	592	
520		600	609	617	625	634	642	650	659	667	675	
521		684	692	700	709	717	725	734	742	750	759	8
522		767	775	784	792	800	809	817	825	834	842	1 | 0.8
523		850	858	867	875	883	892	900	908	917	925	2 | 1.6
524		933	941	950	958	966	975	983	991	999	*008	3 | 2.4
525	72	016	024	032	041	049	057	066	074	082	090	4 | 3.2
526		099	107	115	123	132	140	148	156	165	173	5 | 4.0
527		181	189	198	206	214	222	230	239	247	255	6 | 4.8
528		263	272	280	288	296	304	313	321	329	337	7 | 5.6
529		346	354	362	370	378	387	395	403	411	419	8 | 6.4
530		428	436	444	452	460	469	477	485	493	501	9 | 7.2
531		509	518	526	534	542	550	558	567	575	583	
532		591	599	607	616	624	632	640	648	656	665	
533		673	681	689	697	705	713	722	730	738	746	
534		754	762	770	779	787	795	803	811	819	827	
535		835	843	852	860	868	876	884	892	900	908	
536		916	925	933	941	949	957	965	973	981	989	7
537		997	*006	*014	*022	*030	*038	*046	*054	*062	*070	1 | 0.7
538	73	078	086	094	102	111	119	127	135	143	151	2 | 1.4
539		159	167	175	183	191	199	207	215	223	231	3 | 2.1
540		239	247	255	263	272	280	288	296	304	312	4 | 2.8
541		320	328	336	344	352	360	368	376	384	392	5 | 3.5
542		400	408	416	424	432	440	448	456	464	472	6 | 4.2
543		480	488	496	504	512	520	528	536	544	552	7 | 4.9
544		560	568	576	584	592	600	608	616	624	632	8 | 5.6
545		640	648	656	664	672	679	687	695	703	711	9 | 6.3
546		719	727	735	743	751	759	767	775	783	791	
547		799	807	815	823	830	838	846	854	862	870	
548		878	886	894	902	910	918	926	933	941	949	
549		957	965	973	981	989	997	*005	*013	*020	*028	
550	74	036	044	052	060	068	076	084	092	099	107	
N.	L.	0	1	2	3	4	5	6	7	8	9	Prop. Parts

550–600

N.	L. 0	1	2	3	4	5	6	7	8	9	Prop. Parts
550	74 036	044	052	060	068	076	084	092	099	107	
551	115	123	131	139	147	155	162	170	178	186	
552	194	202	210	218	225	233	241	249	257	265	
553	273	280	288	296	304	312	320	327	335	343	
554	351	359	367	374	382	390	398	406	414	421	
555	429	437	445	453	461	468	476	484	492	500	
556	507	515	523	531	539	547	554	562	570	578	
557	586	593	601	609	617	624	632	640	648	656	
558	663	671	679	687	695	702	710	718	726	733	
559	741	749	757	764	772	780	788	796	803	811	
560	819	827	834	842	850	858	865	873	881	889	8
561	896	904	912	920	927	935	943	950	958	966	1 | 0.8
562	974	981	989	997	*005	*012	*020	*028	*035	*043	2 | 1.6
563	75 051	059	066	074	082	089	097	105	113	120	3 | 2.4
564	128	136	143	151	159	166	174	182	189	197	4 | 3.2
565	205	213	220	228	236	243	251	259	266	274	5 | 4.0
566	282	289	297	305	312	320	328	335	343	351	6 | 4.8
567	358	366	374	381	389	397	404	412	420	427	7 | 5.6
568	435	442	450	458	465	473	481	488	496	504	8 | 6.4
569	511	519	526	534	542	549	557	565	572	580	9 | 7.2
570	587	595	603	610	618	626	633	641	648	656	
571	664	671	679	686	694	702	709	717	724	732	
572	740	747	755	762	770	778	785	793	800	808	
573	815	823	831	838	846	853	861	868	876	884	
574	891	899	906	914	921	929	937	944	952	959	
575	967	974	982	989	997	*005	*012	*020	*027	*035	
576	76 042	050	057	065	072	080	087	095	103	110	
577	118	125	133	140	148	155	163	170	178	185	
578	193	200	208	215	223	230	238	245	253	260	
579	268	275	283	290	298	305	313	320	328	335	
580	343	350	358	365	373	380	388	395	403	410	7
581	418	425	433	440	448	455	462	470	477	485	1 | 0.7
582	492	500	507	515	522	530	537	545	552	559	2 | 1.4
583	567	574	582	589	597	604	612	619	626	634	3 | 2.1
584	641	649	656	664	671	678	686	693	701	708	4 | 2.8
585	716	723	730	738	745	753	760	768	775	782	5 | 3.5
586	790	797	805	812	819	827	834	842	849	856	6 | 4.2
587	864	871	879	886	893	901	908	916	923	930	7 | 4.9
588	938	945	953	960	967	975	982	989	997	*004	8 | 5.6
589	77 012	019	026	034	041	048	056	063	070	078	9 | 6.3
590	085	093	100	107	115	122	129	137	144	151	
591	159	166	173	181	188	195	203	210	217	225	
592	232	240	247	254	262	269	276	283	291	298	
593	305	313	320	327	335	342	349	357	364	371	
594	379	386	393	401	408	415	422	430	437	444	
595	452	459	466	474	481	488	495	503	510	517	
596	525	532	539	546	554	561	568	576	583	590	
597	597	605	612	619	627	634	641	648	656	663	
598	670	677	685	692	699	706	714	721	728	735	
599	743	750	757	764	772	779	786	793	801	808	
600	815	822	830	837	844	851	859	866	873	880	
N.	L. 0	1	2	3	4	5	6	7	8	9	Prop. Parts

600–650

N.	L. 0	1	2	3	4	5	6	7	8	9
600	77 815	822	830	837	844	851	859	866	873	880
601	887	895̄	902	909	916	924	931	938	945	952
602	960	967	974	981	988	996	*003	*010	*017	*025̄
603	78 032	039	046	053	061	068	075̄	082	089	097
604	104	111	118	125	132	140	147	154	161	168
605	176	183	190	197	204	211	219	226	233	240
606	247	254	262	269	276	283	290	297	305̄	312
607	319	326	333	340	347	355̄	362	369	376	383
608	390	398	405̄	412	419	426	433	440	447	455̄
609	462	469	476	483	490	497	504	512	519	526
610	533	540	547	554	561	569	576	583	590	597
611	604	611	618	625	633	640	647	654	661	668
612	675	682	689	696	704	711	718	725̄	732	739
613	746	753	760	767	774	781	789	796	803	810
614	817	824	831	838	845	852	859	866	873	880
615	888	895̄	902	909	916	923	930	937	944	951
616	958	965	972	979	986	993	*000	*007	*014	*021
617	79 029	036	043	050	057	064	071	078	085̄	092
618	099	106	113	120	127	134	141	148	155	162
619	169	176	183	190	197	204	211	218	225	232
620	239	246	253	260	267	274	281	288	295	302
621	309	316	323	330	337	344	351	358	365	372
622	379	386	393	400	407	414	421	428	435̄	442
623	449	456	463	470	477	484	491	498	505̄	511
624	518	525	532	539	546	553	560	567	574	581
625	588	595̄	602	609	616	623	630	637	644	650
626	657	664	671	678	685	692	699	706	713	720
627	727	734	741	748	754	761	768	775	782	789
628	796	803	810	817	824	831	837	844	851	858
629	865	872	879	886	893	900	906	913	920	927
630	934	941	948	955̄	962	969	975	982	989	996
631	80 003	010	017	024	030	037	044	051	058	065̄
632	072	079	085	092	099	106	113	120	127	134
633	140	147	154	161	168	175̄	182	188	195	202
634	209	216	223	229	236	243	250	257	264	271
635	277	284	291	298	305̄	312	318	325	332	339
636	346	353	359	366	373	380	387	393	400	407
637	414	421	428	434	441	448	455̄	462	468	475
638	482	489	496	502	509	516	523	530	536	543
639	550	557	564	570	577	584	591	598	604	611
640	618	625̄	632	638	645	652	659	665	672	679
641	686	693	699	706	713	720	726	733	740	747
642	754	760	767	774	781	787	794	801	808	814
643	821	828	835̄	841	848	855̄	862	868	875	882
644	889	895	902	909	916	922	929	936	943	949
645	956	963	969	976	983	990	996	*003	*010	*017
646	81 023	030	037	043	050	057	064	070	077	084
647	090	097	104	111	117	124	131	137	144	151
648	158	164	171	178	184	191	198	204	211	218
649	224	231	238	245̄	251	258	265̄	271	278	285̄
650	291	298	305̄	311	318	325̄	331	338	345̄	351

| N. | L. 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |

Prop. Parts

	8	7	6
1	0.8	0.7	0.6
2	1.6	1.4	1.2
3	2.4	2.1	1.8
4	3.2	2.8	2.4
5	4.0	3.5	3.0
6	4.8	4.2	3.6
7	5.6	4.9	4.2
8	6.4	5.6	4.8
9	7.2	6.3	5.4

650–700

N.	L. 0	1	2	3	4	5	6	7	8	9	Prop. Parts
650	81 291	298	305̄	311	318	325̄	331	338	345̄	351	
651	358	365̄	371	378	385̄	391	398	405̄	411	418	
652	425̄	431	438	445̄	451	458	465̄	471	478	485̄	
653	491	498	505̄	511	518	525̄	531	538	544	551	
654	558	564	571	578	584	591	598	604	611	617	
655	624	631	637	644	651	657	664	671	677	684	
656	690	697	704	710	717	723	730	737	743	750	
657	757	763	770	776	783	790	796	803	809	816	
658	823	829	836	842	849	856	862	869	875	882	
659	889	895	902	908	915̄	921	928	935̄	941	948	
660	954	961	968	974	981	987	994	*000	*007	*014	
661	82 020	027	033	040	046	053	060	066	073	079	
662	086	092	099	105	112	119	125	132	138	145̄	
663	151	158	164	171	178	184	191	197	204	210	
664	217	223	230	236	243	249	256	263	269	276	
665	282	289	295	302	308	315̄	321	328	334	341	
666	347	354	360	367	373	380	387	393	400	406	
667	413	419	426	432	439	445	452	458	465̄	471	
668	478	484	491	497	504	510	517	523	530	536	
669	543	549	556	562	569	575	582	588	595̄	601	
670	607	614	620	627	633	640	646	653	659	666	
671	672	679	685	692	698	705̄	711	718	724	730	
672	737	743	750	756	763	769	776	782	789	795	
673	802	808	814	821	827	834	840	847	853	860	
674	866	872	879	885	892	898	905̄	911	918	924	
675	930	937	943	950	956	963	969	975	982	988	
676	995̄	*001	*008	*014	*020	*027	*033	*040	*046	*052	
677	83 059	065	072	078	085̄	091	097	104	110	117	
678	123	129	136	142	149	155̄	161	168	174	181	
679	187	193	200	206	213	219	225	232	238	245̄	
680	251	257	264	270	276	283	289	296	302	308	
681	315̄	321	327	334	340	347	353	359	366	372	
682	378	385̄	391	398	404	410	417	423	429	436	
683	442	448	455̄	461	467	474	480	487	493	499	
684	506	512	518	525̄	531	537	544	550	556	563	
685	569	575	582	588	594	601	607	613	620	626	
686	632	639	645	651	658	664	670	677	683	689	
687	696	702	708	715̄	721	727	734	740	746	753	
688	759	765	771	778	784	790	797	803	809	816	
689	822	828	835̄	841	847	853	860	866	872	879	
690	885̄	891	897	904	910	916	923	929	935	942	
691	948	954	960	967	973	979	985	992	998	*004	
692	84 011	017	023	029	036	042	048	055̄	061	067	
693	073	080	086	092	098	105̄	111	117	123	130	
694	136	142	148	155̄	161	167	173	180	186	192	
695	198	205̄	211	217	223	230	236	242	248	255̄	
696	261	267	273	280	286	292	298	305̄	311	317	
697	323	330	336	342	348	354	361	367	373	379	
698	386	392	398	404	410	417	423	429	435	442	
699	448	454	460	466	473	479	485̄	491	497	504	
700	510	516	522	528	535̄	541	547	553	559	566	
N.	L. 0	1	2	3	4	5	6	7	8	9	Prop. Parts

Prop. Parts:

	7
1	0.7
2	1.4
3	2.1
4	2.8
5	3.5
6	4.2
7	4.9
8	5.6
9	6.3

	6
1	0.6
2	1.2
3	1.8
4	2.4
5	3.0
6	3.6
7	4.2
8	4.8
9	5.4

700–750

N.	L. 0	1	2	3	4	5	6	7	8	9
700	84 510	516	522	528	535	541	547	553	559	566
701	572	578	584	590	597	603	609	615	621	628
702	634	640	646	652	658	665	671	677	683	689
703	696	702	708	714	720	726	733	739	745	751
704	757	763	770	776	782	788	794	800	807	813
705	819	825	831	837	844	850	856	862	868	874
706	880	887	893	899	905	911	917	924	930	936
707	942	948	954	960	967	973	979	985	991	997
708	85 003	009	016	022	028	034	040	046	052	058
709	065	071	077	083	089	095	101	107	114	120
710	126	132	138	144	150	156	163	169	175	181
711	187	193	199	205	211	217	224	230	236	242
712	248	254	260	266	272	278	285	291	297	303
713	309	315	321	327	333	339	345	352	358	364
714	370	376	382	388	394	400	406	412	418	425
715	431	437	443	449	455	461	467	473	479	485
716	491	497	503	509	516	522	528	534	540	546
717	552	558	564	570	576	582	588	594	600	606
718	612	618	625	631	637	643	649	655	661	667
719	673	679	685	691	697	703	709	715	721	727
720	733	739	745	751	757	763	769	775	781	788
721	794	800	806	812	818	824	830	836	842	848
722	854	860	866	872	878	884	890	896	902	908
723	914	920	926	932	938	944	950	956	962	968
724	974	980	986	992	998	*004	*010	*016	*022	*028
725	86 034	040	046	052	058	064	070	076	082	088
726	094	100	106	112	118	124	130	136	141	147
727	153	159	165	171	177	183	189	195	201	207
728	213	219	225	231	237	243	249	255	261	267
729	273	279	285	291	297	303	308	314	320	326
730	332	338	344	350	356	362	368	374	380	386
731	392	398	404	410	415	421	427	433	439	445
732	451	457	463	469	475	481	487	493	499	504
733	510	516	522	528	534	540	546	552	558	564
734	570	576	581	587	593	599	605	611	617	623
735	629	635	641	646	652	658	664	670	676	682
736	688	694	700	705	711	717	723	729	735	741
737	747	753	759	764	770	776	782	788	794	800
738	806	812	817	823	829	835	841	847	853	859
739	864	870	876	882	888	894	900	906	911	917
740	923	929	935	941	947	953	958	964	970	976
741	982	988	994	999	*005	*011	*017	*023	*029	*035
742	87 040	046	052	058	064	070	075	081	087	093
743	099	105	111	116	122	128	134	140	146	151
744	157	163	169	175	181	186	192	198	204	210
745	216	221	227	233	239	245	251	256	262	268
746	274	280	286	291	297	303	309	315	320	326
747	332	338	344	349	355	361	367	373	379	384
748	390	396	402	408	413	419	425	431	437	442
749	448	454	460	466	471	477	483	489	495	500
750	506	512	518	523	529	535	541	547	552	558
N.	L. 0	1	2	3	4	5	6	7	8	9

Prop. Parts

	7
1	0.7
2	1.4
3	2.1
4	2.8
5	3.5
6	4.2
7	4.9
8	5.6
9	6.3

	6
1	0.6
2	1.2
3	1.8
4	2.4
5	3.0
6	3.6
7	4.2
8	4.8
9	5.4

	5
1	0.5
2	1.0
3	1.5
4	2.0
5	2.5
6	3.0
7	3.5
8	4.0
9	4.5

Prop. Parts

750–800

N.	L. 0	1	2	3	4	5	6	7	8	9
750	87 506	512	518	523	529	535	541	547	552	558
751	564	570	576	581	587	593	599	604	610	616
752	622	628	633	639	645	651	656	662	668	674
753	679	685	691	697	703	708	714	720	726	731
754	737	743	749	754	760	766	772	777	783	789
755	795	800	806	812	818	823	829	835	841	846
756	852	858	864	869	875	881	887	892	898	904
757	910	915	921	927	933	938	944	950	955	961
758	967	973	978	984	990	996	*001	*007	*013	*018
759	88 024	030	036	041	047	053	058	064	070	076
760	081	087	093	098	104	110	116	121	127	133
761	138	144	150	156	161	167	173	178	184	190
762	195	201	207	213	218	224	230	235	241	247
763	252	258	264	270	275	281	287	292	298	304
764	309	315	321	326	332	338	343	349	355	360
765	366	372	377	383	389	395	400	406	412	417
766	423	429	434	440	446	451	457	463	468	474
767	480	485	491	497	502	508	513	519	525	530
768	536	542	547	553	559	564	570	576	581	587
769	593	598	604	610	615	621	627	632	638	643
770	649	655	660	666	672	677	683	689	694	700
771	705	711	717	722	728	734	739	745	750	756
772	762	767	773	779	784	790	795	801	807	812
773	818	824	829	835	840	846	852	857	863	868
774	874	880	885	891	897	902	908	913	919	925
775	930	936	941	947	953	958	964	969	975	981
776	986	992	997	*003	*009	*014	*020	*025	*031	*037
777	89 042	048	053	059	064	070	076	081	087	092
778	098	104	109	115	120	126	131	137	143	148
779	154	159	165	170	176	182	187	193	198	204
780	209	215	221	226	232	237	243	248	254	260
781	265	271	276	282	287	293	298	304	310	315
782	321	326	332	337	343	348	354	360	365	371
783	376	382	387	393	398	404	409	415	421	426
784	432	437	443	448	454	459	465	470	476	481
785	487	492	498	504	509	515	520	526	531	537
786	542	548	553	559	564	570	575	581	586	592
787	597	603	609	614	620	625	631	636	642	647
788	653	658	664	669	675	680	686	691	697	702
789	708	713	719	724	730	735	741	746	752	757
790	763	768	774	779	785	790	796	801	807	812
791	818	823	829	834	840	845	851	856	862	867
792	873	878	883	889	894	900	905	911	916	922
793	927	933	938	944	949	955	960	966	971	977
794	982	988	993	998	*004	*009	*015	*020	*026	*031
795	90 037	042	048	053	059	064	069	075	080	086
796	091	097	102	108	113	119	124	129	135	140
797	146	151	157	162	168	173	179	184	189	195
798	200	206	211	217	222	227	233	238	244	249
799	255	260	266	271	276	282	287	293	298	304
800	309	314	320	325	331	336	342	347	352	358
N.	L. 0	1	2	3	4	5	6	7	8	9

Prop. Parts

	6
1	0.6
2	1.2
3	1.8
4	2.4
5	3.0
6	3.6
7	4.2
8	4.8
9	5.4

	5
1	0.5
2	1.0
3	1.5
4	2.0
5	2.5
6	3.0
7	3.5
8	4.0
9	4.5

Prop. Parts

800–850

N.	L. 0	1	2	3	4	5	6	7	8	9	Prop. Parts
800	90 309	314	320	325	331	336	342	347	352	358	
801	363	369	374	380	38$\bar{5}$	390	396	401	407	412	
802	417	423	428	434	439	44$\bar{5}$	4$\bar{5}$0	455	461	466	
803	472	477	482	488	493	499	504	509	51$\bar{5}$	520	
804	526	531	536	542	547	553	558	563	569	574	
805	580	58$\bar{5}$	590	596	601	607	612	617	623	628	
806	634	639	644	6$\bar{5}$0	655	660	666	671	677	682	
807	687	693	698	703	709	714	720	725	730	736	
808	741	747	752	757	763	768	773	779	784	789	
809	79$\bar{5}$	800	806	811	816	822	827	832	838	843	
810	849	854	859	86$\bar{5}$	870	875	881	886	891	897	
811	902	907	913	918	924	929	934	940	94$\bar{5}$	950	**6**
812	956	961	966	972	977	982	988	993	998	*004	1 | 0.6
813	91 009	014	020	025	030	036	041	046	052	057	2 | 1.2
814	062	068	073	078	084	089	094	100	105	110	3 | 1.8
815	116	121	126	132	137	142	148	153	158	164	4 | 2.4
816	169	174	180	18$\bar{5}$	190	196	201	206	212	217	5 | 3.0
817	222	228	233	238	243	249	254	259	26$\bar{5}$	270	6 | 3.6
818	275	281	286	291	297	302	307	312	318	323	7 | 4.2
819	328	334	339	344	3$\bar{5}$0	35$\bar{5}$	360	365	371	376	8 | 4.8
820	381	387	392	397	403	408	413	418	424	429	9 | 5.4
821	434	440	44$\bar{5}$	450	455	461	466	471	477	482	
822	487	492	498	503	508	514	519	524	529	53$\bar{5}$	
823	540	545	551	556	561	566	572	577	582	587	
824	593	598	603	609	614	619	624	630	63$\bar{5}$	640	
825	645	651	656	661	666	672	677	682	687	693	
826	698	703	709	714	719	724	730	73$\bar{5}$	740	745	
827	751	756	761	766	772	777	782	787	793	798	
828	803	808	814	819	824	829	834	840	84$\bar{5}$	850	
829	855	861	866	871	876	882	887	892	897	903	
830	908	913	918	924	929	934	939	944	9$\bar{5}$0	95$\bar{5}$	
831	960	965	971	976	981	986	991	997	*002	*007	**5**
832	92 012	018	023	028	033	038	044	049	054	059	1 | 0.5
833	06$\bar{5}$	070	07$\bar{5}$	080	085	091	096	101	106	111	2 | 1.0
834	117	122	127	132	137	143	148	153	158	163	3 | 1.5
835	169	174	179	184	189	19$\bar{5}$	200	205	210	215	4 | 2.0
836	221	226	231	236	241	247	252	257	262	267	5 | 2.5
837	273	278	283	288	293	298	304	309	314	319	6 | 3.0
838	324	330	33$\bar{5}$	340	345	350	355	361	366	371	7 | 3.5
839	376	381	387	392	397	402	407	412	418	423	8 | 4.0
840	428	433	438	443	449	454	459	464	469	474	9 | 4.5
841	480	48$\bar{5}$	490	495	500	505	511	516	521	526	
842	531	536	542	547	552	557	562	567	572	578	
843	583	588	593	598	603	609	614	619	624	629	
844	634	639	64$\bar{5}$	6$\bar{5}$0	65$\bar{5}$	660	665	670	675	681	
845	686	691	696	701	706	711	716	722	727	732	
846	737	742	747	752	758	763	768	773	778	783	
847	788	793	799	804	809	814	819	824	829	834	
848	840	84$\bar{5}$	8$\bar{5}$0	85$\bar{5}$	860	865	870	875	881	886	
849	891	896	901	906	911	916	921	927	932	937	
850	942	947	952	957	962	967	973	978	983	988	
N.	L. 0	1	2	3	4	5	6	7	8	9	Prop. Parts

850–900

N.	L. 0	1	2	3	4	5	6	7	8	9
850	92 942	947	952	957	962	967	973	978	983	988
851	993	998	*003	*008	*013	*018	*024	*029	*034	*039
852	93 044	049	054	059	064	069	075	080	085	090
853	095	100	105	110	115	120	125	131	136	141
854	146	151	156	161	166	171	176	181	186	192
855	197	202	207	212	217	222	227	232	237	242
856	247	252	258	263	268	273	278	283	288	293
857	298	303	308	313	318	323	328	334	339	344
858	349	354	359	364	369	374	379	384	389	394
859	399	404	409	414	420	425	430	435	440	445
860	450	455	460	465	470	475	480	485	490	495
861	500	505	510	515	520	526	531	536	541	546
862	551	556	561	566	571	576	581	586	591	596
863	601	606	611	616	621	626	631	636	641	646
864	651	656	661	666	671	676	682	687	692	697
865	702	707	712	717	722	727	732	737	742	747
866	752	757	762	767	772	777	782	787	792	797
867	802	807	812	817	822	827	832	837	842	847
868	852	857	862	867	872	877	882	887	892	897
869	902	907	912	917	922	927	932	937	942	947
870	952	957	962	967	972	977	982	987	992	997
871	94 002	007	012	017	022	027	032	037	042	047
872	052	057	062	067	072	077	082	086	091	096
873	101	106	111	116	121	126	131	136	141	146
874	151	156	161	166	171	176	181	186	191	196
875	201	206	211	216	221	226	231	236	240	245
876	250	255	260	265	270	275	280	285	290	295
877	300	305	310	315	320	325	330	335	340	345
878	349	354	359	364	369	374	379	384	389	394
879	399	404	409	414	419	424	429	433	438	443
880	448	453	458	463	468	473	478	483	488	493
881	498	503	507	512	517	522	527	532	537	542
882	547	552	557	562	567	571	576	581	586	591
883	596	601	606	611	616	621	626	630	635	640
884	645	650	655	660	665	670	675	680	685	689
885	694	699	704	709	714	719	724	729	734	738
886	743	748	753	758	763	768	773	778	783	787
887	792	797	802	807	812	817	822	827	832	836
888	841	846	851	856	861	866	871	876	880	885
889	890	895	900	905	910	915	919	924	929	934
890	939	944	949	954	959	963	968	973	978	983
891	988	993	998	*002	*007	*012	*017	*022	*027	*032
892	95 036	041	046	051	056	061	066	071	075	080
893	085	090	095	100	105	109	114	119	124	129
894	134	139	143	148	153	158	163	168	173	177
895	182	187	192	197	202	207	211	216	221	226
896	231	236	240	245	250	255	260	265	270	274
897	279	284	289	294	299	303	308	313	318	323
898	328	332	337	342	347	352	357	361	366	371
899	376	381	386	390	395	400	405	410	415	419
900	424	429	434	439	444	448	453	458	463	468
N.	L. 0	1	2	3	4	5	6	7	8	9

Prop. Parts

	6	5	4
1	0.6	0.5	0.4
2	1.2	1.0	0.8
3	1.8	1.5	1.2
4	2.4	2.0	1.6
5	3.0	2.5	2.0
6	3.6	3.0	2.4
7	4.2	3.5	2.8
8	4.8	4.0	3.2
9	5.4	4.5	3.6

900–950

N.	L. 0	1	2	3	4	5	6	7	8	9	Prop. Parts
900	95 424	429	434	439	444	448	453	458	463	468	
901	472	477	482	487	492	497	501	506	511	516	
902	521	525	530	535	540	545	550	554	559	564	
903	569	574	578	583	588	593	598	602	607	612	
904	617	622	626	631	636	641	646	650	655	660	
905	665	670	674	679	684	689	694	698	703	708	
906	713	718	722	727	732	737	742	746	751	756	
907	761	766	770	775	780	785	789	794	799	804	
908	809	813	818	823	828	832	837	842	847	852	
909	856	861	866	871	875	880	885	890	895	899	
910	904	909	914	918	923	928	933	938	942	947	
911	952	957	961	966	971	976	980	985	990	995	5
912	999	*004	*009	*014	*019	*023	*028	*033	*038	*042	1 | 0.5
913	96 047	052	057	061	066	071	076	080	085	090	2 | 1.0
914	095	099	104	109	114	118	123	128	133	137	3 | 1.5
915	142	147	152	156	161	166	171	175	180	185	4 | 2.0
916	190	194	199	204	209	213	218	223	227	232	5 | 2.5
917	237	242	246	251	256	261	265	270	275	280	6 | 3.0
918	284	289	294	298	303	308	313	317	322	327	7 | 3.5
919	332	336	341	346	350	355	360	365	369	374	8 | 4.0
920	379	384	388	393	398	402	407	412	417	421	9 | 4.5
921	426	431	435	440	445	450	454	459	464	468	
922	473	478	483	487	492	497	501	506	511	515	
923	520	525	530	534	539	544	548	553	558	562	
924	567	572	577	581	586	591	595	600	605	609	
925	614	619	624	628	633	638	642	647	652	656	
926	661	666	670	675	680	685	689	694	699	703	
927	708	713	717	722	727	731	736	741	745	750	
928	755	759	764	769	774	778	783	788	792	797	
929	802	806	811	816	820	825	830	834	839	844	
930	848	853	858	862	867	872	876	881	886	890	
931	895	900	904	909	914	918	923	928	932	937	4
932	942	946	951	956	960	965	970	974	979	984	1 | 0.4
933	988	993	997	*002	*007	*011	*016	*021	*025	*030	2 | 0.8
934	97 035	039	044	049	053	058	063	067	072	077	3 | 1.2
935	081	086	090	095	100	104	109	114	118	123	4 | 1.6
936	128	132	137	142	146	151	155	160	165	169	5 | 2.0
937	174	179	183	188	192	197	202	206	211	216	6 | 2.4
938	220	225	230	234	239	243	248	253	257	262	7 | 2.8
939	267	271	276	280	285	290	294	299	304	308	8 | 3.2
940	313	317	322	327	331	336	340	345	350	354	9 | 3.
941	359	364	368	373	377	382	387	391	396	400	
942	405	410	414	419	424	428	433	437	442	447	
943	451	456	460	465	470	474	479	483	488	493	
944	497	502	506	511	516	520	525	529	534	539	
945	543	548	552	557	562	566	571	575	580	585	
946	589	594	598	603	607	612	617	621	626	630	
947	635	640	644	649	653	658	663	667	672	676	
948	681	685	690	695	699	704	708	713	717	722	
949	727	731	736	740	745	749	754	759	763	768	
950	772	777	782	786	791	795	800	804	809	813	
N.	L. 0	1	2	3	4	5	6	7	8	9	Prop. Parts

950–1000

N.	L. 0	1	2	3	4	5	6	7	8	9	Prop. Parts
950	97 772	777	782	786	791	795	800	804	809	813	
951	818	823	827	832	836	841	845	850	$85\bar{5}$	859	
952	864	868	873	877	882	886	891	896	900	$90\bar{5}$	
953	909	914	918	923	928	932	937	941	946	950	
954	$95\bar{5}$	959	964	968	973	978	982	987	991	996	
955	98 000	$00\bar{5}$	009	014	019	023	028	032	037	041	
956	046	050	$05\bar{5}$	059	064	068	073	078	082	087	
957	091	096	100	$10\bar{5}$	109	114	118	123	127	132	
958	137	141	146	150	$15\bar{5}$	159	164	168	173	177	
959	182	186	191	195	200	204	209	214	218	223	
960	227	232	236	241	245	250	254	259	263	268	**5**
961	272	277	281	286	290	$29\bar{5}$	299	304	308	313	1 | 0.5
962	318	322	327	331	336	340	$34\bar{5}$	349	354	358	2 | 1.0
963	363	367	372	376	381	385	390	394	399	403	3 | 1.5
964	408	412	417	421	426	430	$43\bar{5}$	439	444	448	4 | 2.0
965	453	457	462	466	471	475	480	484	489	493	5 | 2.5
966	498	502	507	511	516	520	$52\bar{5}$	529	534	538	6 | 3.0
967	543	547	552	556	561	565	570	574	579	583	7 | 3.5
968	588	592	597	601	605	610	614	619	623	628	8 | 4.0
969	632	637	641	646	650	$65\bar{5}$	659	664	668	673	9 | 4.5
970	677	682	686	691	695	700	704	709	713	717	
971	722	726	731	735	740	744	749	753	758	762	
972	767	771	776	780	784	789	793	798	802	807	
973	811	816	820	$82\bar{5}$	829	834	838	843	847	851	
974	856	860	$86\bar{5}$	869	874	878	883	887	892	896	
975	900	$90\bar{5}$	909	914	918	923	927	932	936	941	
976	$94\bar{5}$	949	954	958	963	967	972	976	981	985	
977	989	994	998	*003	*007	*012	*016	*021	*025	*029	
978	99 034	038	043	047	052	056	061	$06\bar{5}$	069	074	
979	078	083	087	092	096	100	$10\bar{5}$	109	114	118	
980	123	127	131	136	140	$14\bar{5}$	149	154	158	162	**4**
981	167	171	176	180	$18\bar{5}$	189	193	198	202	207	1 | 0.4
982	211	216	220	224	229	233	238	242	247	251	2 | 0.8
983	255	260	264	269	273	277	282	286	291	295	3 | 1.2
984	300	304	308	313	317	322	326	330	$33\bar{5}$	339	4 | 1.6
985	344	348	352	357	361	366	370	374	379	383	5 | 2.0
986	388	392	396	401	405	410	414	419	423	427	6 | 2.4
987	432	436	441	$44\bar{5}$	449	454	458	463	467	471	7 | 2.8
988	476	480	484	489	493	498	502	506	511	515	8 | 3.2
989	520	524	528	533	537	542	546	550	$55\bar{5}$	559	9 | 3.6
990	564	568	572	577	581	585	590	594	599	603	
991	607	612	616	621	$62\bar{5}$	629	634	638	642	647	
992	651	656	660	664	669	673	677	682	686	691	
993	$69\bar{5}$	699	704	708	712	717	721	726	730	734	
994	739	743	747	752	756	760	$76\bar{5}$	769	774	778	
995	782	787	791	795	800	804	808	813	817	822	
996	826	830	$83\bar{5}$	839	843	848	852	856	861	865	
997	870	874	878	883	887	891	896	900	904	909	
998	913	917	922	926	930	$93\bar{5}$	939	944	948	952	
999	957	961	965	970	974	978	983	987	991	996	
1000	00 000	004	009	013	017	022	026	030	$03\bar{5}$	039	
N.	L. 0	1	2	3	4	5	6	7	8	9	Prop. Parts

1000–1050

N.	L.	0	1	2	3	4	5	6	7	8	9
1000	000	0000	0434	0869	1303	1737	2171	2605	3039	3473	3907
1001		4341	4775	5208	5642	6076	6510	6943	7377	7810	8244
1002		8677	9111	9544	9977	*0411	*0844	*1277	*1710	*2143	*2576
1003	001	3009	3442	3875	4308	4741	5174	5607	6039	6472	6905
1004		7337	7770	8202	8635	9067	9499	9932	*0364	*0796	*1228
1005	002	1661	2093	2525	2957	3389	3821	4253	4685	5116	5548
1006		5980	6411	6843	7275	7706	8138	8569	9001	9432	9863
1007	003	0295	0726	1157	1588	2019	2451	2882	3313	3744	4174
1008		4605	5036	5467	5898	6328	6759	7190	7620	8051	8481
1009		8912	9342	9772	*0203	*0633	*1063	*1493	*1924	*2354	*2784
1010	004	3214	3644	4074	4504	4933	5363	5793	6223	6652	7082
1011		7512	7941	8371	8800	9229	9659	*0088	*0517	*0947	*1376
1012	005	1805	2234	2663	3092	3521	3950	4379	4808	5237	5666
1013		6094	6523	6952	7380	7809	8238	8666	9094	9523	9951
1014	006	0380	0808	1236	1664	2092	2521	2949	3377	3805	4233
1015		4660	5088	5516	5944	6372	6799	7227	7655	8082	8510
1016		8937	9365	9792	*0219	*0647	*1074	*1501	*1928	*2355	*2782
1017	007	3210	3637	4064	4490	4917	5344	5771	6198	6624	7051
1018		7478	7904	8331	8757	9184	9610	*0037	*0463	*0889	*1316
1019	008	1742	2168	2594	3020	3446	3872	4298	4724	5150	5576
1020		6002	6427	6853	7279	7704	8130	8556	8981	9407	9832
1021	009	0257	0683	1108	1533	1959	2384	2809	3234	3659	4084
1022		4509	4934	5359	5784	6208	6633	7058	7483	7907	8332
1023		8756	9181	9605	*0030	*0454	*0878	*1303	*1727	*2151	*2575
1024	010	3000	3424	3848	4272	4696	5120	5544	5967	6391	6815
1025		7239	7662	8086	8510	8933	9357	9780	*0204	*0627	*1050
1026	011	1474	1897	2320	2743	3166	3590	4013	4436	4859	5282
1027		5704	6127	6550	6973	7396	7818	8241	8664	9086	9509
1028		9931	*0354	*0776	*1198	*1621	*2043	*2465	*2887	*3310	*3732
1029	012	4154	4576	4998	5420	5842	6264	6685	7107	7529	7951
1030		8372	8794	9215	9637	*0059	*0480	*0901	*1323	*1744	*2165
1031	013	2587	3008	3429	3850	4271	4692	5113	5534	5955	6376
1032		6797	7218	7639	8059	8480	8901	9321	9742	*0162	*0583
1033	014	1003	1424	1844	2264	2685	3105	3525	3945	4365	4785
1034		5205	5625	6045	6465	6885	7305	7725	8144	8564	8984
1035		9403	9823	*0243	*0662	*1082	*1501	*1920	*2340	*2759	*3178
1036	015	3598	4017	4436	4855	5274	5693	6112	6531	6950	7369
1037		7788	8206	8625	9044	9462	9881	*0300	*0718	*1137	*1555
1038	016	1974	2392	2810	3229	3647	4065	4483	4901	5319	5737
1039		6155	6573	6991	7409	7827	8245	8663	9080	9498	9916
1040	017	0333	0751	1168	1586	2003	2421	2838	3256	3673	4090
1041		4507	4924	5342	5759	6176	6593	7010	7427	7844	8260
1042		8677	9094	9511	9927	*0344	*0761	*1177	*1594	*2010	*2427
1043	018	2843	3259	3676	4092	4508	4925	5341	5757	6173	6589
1044		7005	7421	7837	8253	8669	9084	9500	9916	*0332	*0747
1045	019	1163	1578	1994	2410	2825	3240	3656	4071	4486	4902
1046		5317	5732	6147	6562	6977	7392	7807	8222	8637	9052
1047		9467	9882	*0296	*0711	*1126	*1540	*1955	*2369	*2784	*3198
1048	020	3613	4027	4442	4856	5270	5684	6099	6513	6927	7341
1049		7755	8169	8583	8997	9411	9824	*0238	*0652	*1066	*1479
1050	021	1893	2307	2720	3134	3547	3961	4374	4787	5201	5614
N.	L.	0	1	2	3	4	5	6	7	8	9

1050–1100

N.	L.	0	1	2	3	4	5	6	7	8	9
1050	021	1893	2307	2720	3134	3547	3961	4374	4787	5201	5614
1051		6027	6440	6854	7267	7680	8093	8506	8919	9332	9745
1052	022	0157	0570	0983	1396	1808	2221	2634	3046	3459	3871
1053		4284	4696	5109	5521	5933	6345	6758	7170	7582	7994
1054		8406	8818	9230	9642	*0054	*0466	*0878	*1289	*1701	*2113
1055	023	2525	2936	3348	3759	4171	4582	4994	5405	5817	6228
1056		6639	7050	7462	7873	8284	8695	9106	9517	9928	*0339
1057	024	0750	1161	1572	1982	2393	2804	3214	3625	4036	4446
1058		4857	5267	5678	6088	6498	6909	7319	7729	8139	8549
1059		8960	9370	9780	*0190	*0600	*1010	*1419	*1829	*2239	*2649
1060	025	3059	3468	3878	4288	4697	5107	5516	5926	6335	6744
1061		7154	7563	7972	8382	8791	9200	9609	*0018	*0427	*0836
1062	026	1245	1654	2063	2472	2881	3289	3698	4107	4515	4924
1063		5333	5741	6150	6558	6967	7375	7783	8192	8600	9008
1064		9416	9824	*0233	*0641	*1049	*1457	*1865	*2273	*2680	*3088
1065	027	3496	3904	4312	4719	5127	5535	5942	6350	6757	7165
1066		7572	7979	8387	8794	9201	9609	*0016	*0423	*0830	*1237
1067	028	1644	2051	2458	2865	3272	3679	4086	4492	4899	5306
1068		5713	6119	6526	6932	7339	7745	8152	8558	8964	9371
1069		9777	*0183	*0590	*0996	*1402	*1808	*2214	*2620	*3026	*3432
1070	029	3838	4244	4649	5055	5461	5867	6272	6678	7084	7489
1071		7895	8300	8706	9111	9516	9922	*0327	*0732	*1138	*1543
1072	030	1948	2353	2758	3163	3568	3973	4378	4783	5188	5592
1073		5997	6402	6807	7211	7616	8020	8425	8830	9234	9638
1074	031	0043	0447	0851	1256	1660	2064	2468	2872	3277	3681
1075		4085	4489	4893	5296	5700	6104	6508	6912	7315	7719
1076		8123	8526	8930	9333	9737	*0140	*0544	*0947	*1350	*1754
1077	032	2157	2560	2963	3367	3770	4173	4576	4979	5382	5785
1078		6188	6590	6993	7396	7799	8201	8604	9007	9409	9812
1079	033	0214	0617	1019	1422	1824	2226	2629	3031	3433	3835
1080		4238	4640	5042	5444	5846	6248	6650	7052	7453	7855
1081		8257	8659	9060	9462	9864	*0265	*0667	*1068	*1470	*1871
1082	034	2273	2674	3075	3477	3878	4279	4680	5081	5482	5884
1083		6285	6686	7087	7487	7888	8289	8690	9091	9491	9892
1084	035	0293	0693	1094	1495	1895	2296	2696	3096	3497	3897
1085		4297	4698	5098	5498	5898	6298	6698	7098	7498	7898
1086		8298	8698	9098	9498	9898	*0297	*0697	*1097	*1496	*1896
1087	036	2295	2695	3094	3494	3893	4293	4692	5091	5491	5890
1088		6289	6688	7087	7486	7885	8284	8683	9082	9481	9880
1089	037	0279	0678	1076	1475	1874	2272	2671	3070	3468	3867
1090		4265	4663	5062	5460	5858	6257	6655	7053	7451	7849
1091		8248	8646	9044	9442	9839	*0237	*0635	*1033	*1431	*1829
1092	038	2226	2624	3022	3419	3817	4214	4612	5009	5407	5804
1093		6202	6599	6996	7393	7791	8188	8585	8982	9379	9776
1094	039	0173	0570	0967	1364	1761	2158	2554	2951	3348	3745
1095		4141	4538	4934	5331	5727	6124	6520	6917	7313	7709
1096		8106	8502	8898	9294	9690	*0086	*0482	*0878	*1274	*1670
1097	040	2066	2462	2858	3254	3650	4045	4441	4837	5232	5628
1098		6023	6419	6814	7210	7605	8001	8396	8791	9187	9582
1099		9977	*0372	*0767	*1162	*1557	*1952	*2347	*2742	*3137	*3532
1100	041	3927	4322	4716	5111	5506	5900	6295	6690	7084	7479
N.	L.	0	1	2	3	4	5	6	7	8	9

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